Neutrino oscillograms of the Earth, $\not{C}\not{P}$ and future atmospheric neutrino experiments

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High-E neutrino oscillations in the Earth

In collaboration with Michele Maltoni and Alexei Smirnov

A sequel to

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JHEP 0705 (2007) 077 [hep-ph/0612285]

Work in progress

Atmospheric neutrinos:

- Atmospheric neutrino experiments led to the first unambiguous evidence for neutrino oscillations
- About a half of atmospheric neutrinos traverse the Earth on their way to the detector
- Matter can strongly affect ν oscillations inside the Earth through the MSW and parametric resonance effects
- Neutrino oscillograms of the Earth carry a wealth of information both on neutrinos and the Earth

Neutrino oscillograms of the Earth

Contours of equal osc. probabilities in (Θ_ν, E_ν) plane

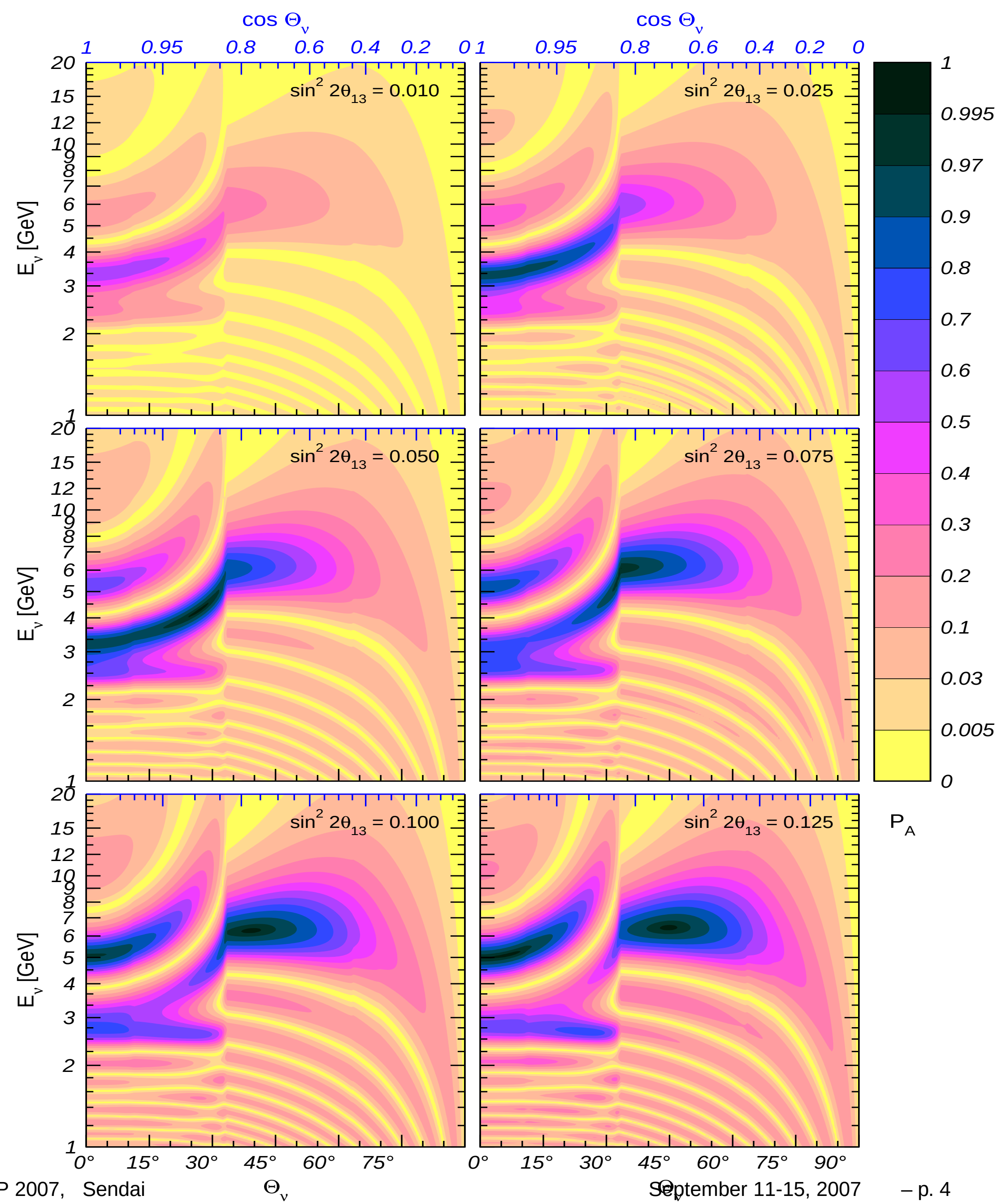
Θ_{13} - dependence of $P_2 \Rightarrow$

P_2 – effective 2f transition probability $(\Delta m^2_{\rm sol} \to 0)$

$$P_{e\mu} = s_{23}^2 P_2$$

$$P_{e\tau} = c_{23}^2 P_2$$

(E.A., Maltoni & Smirnov, 2006)

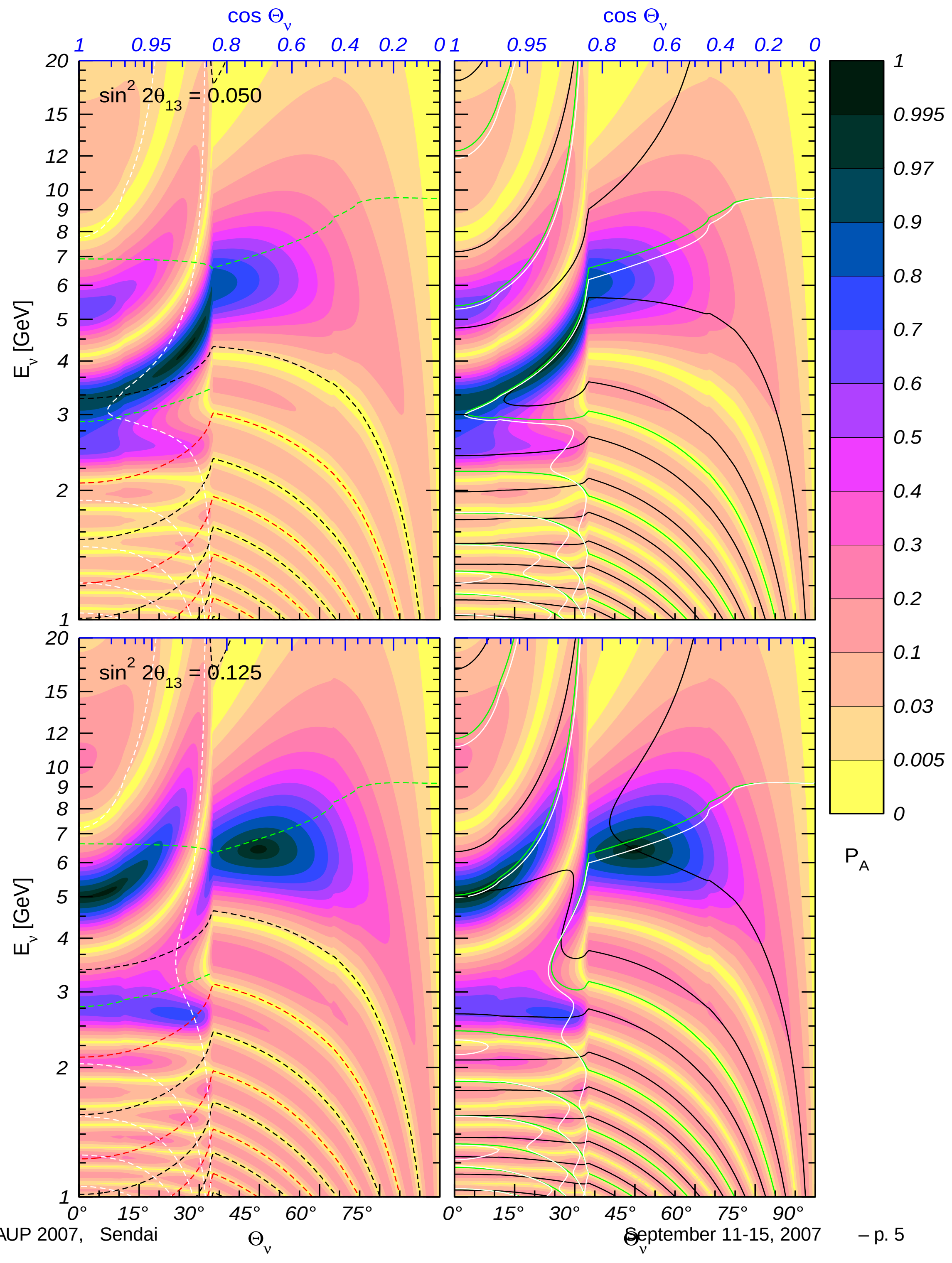


In the right panels:

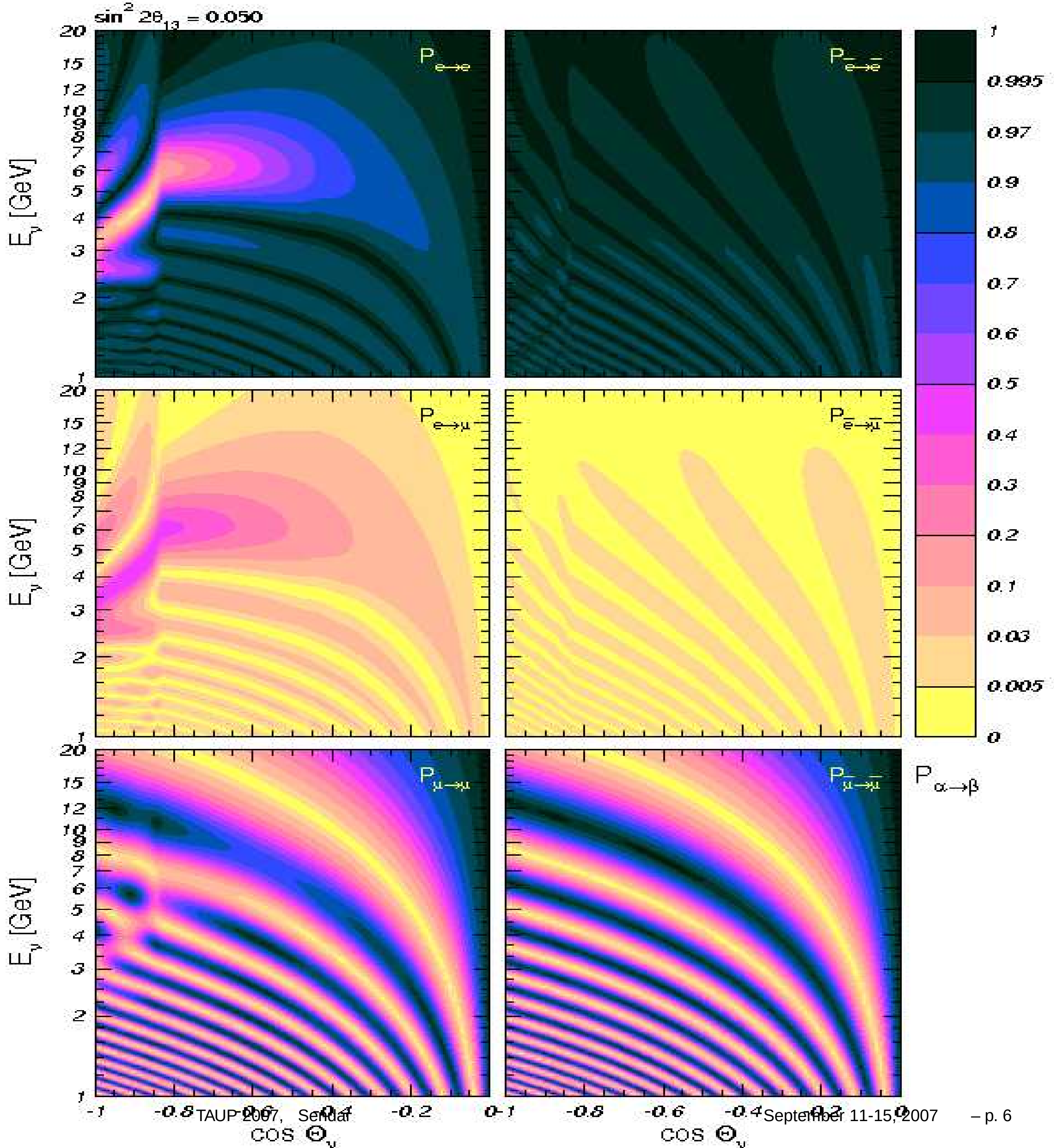
– alignment (collin.) cond.

– generalized res. cond. ($\mathrm{Im}\,\alpha^{(2)} = 0$)

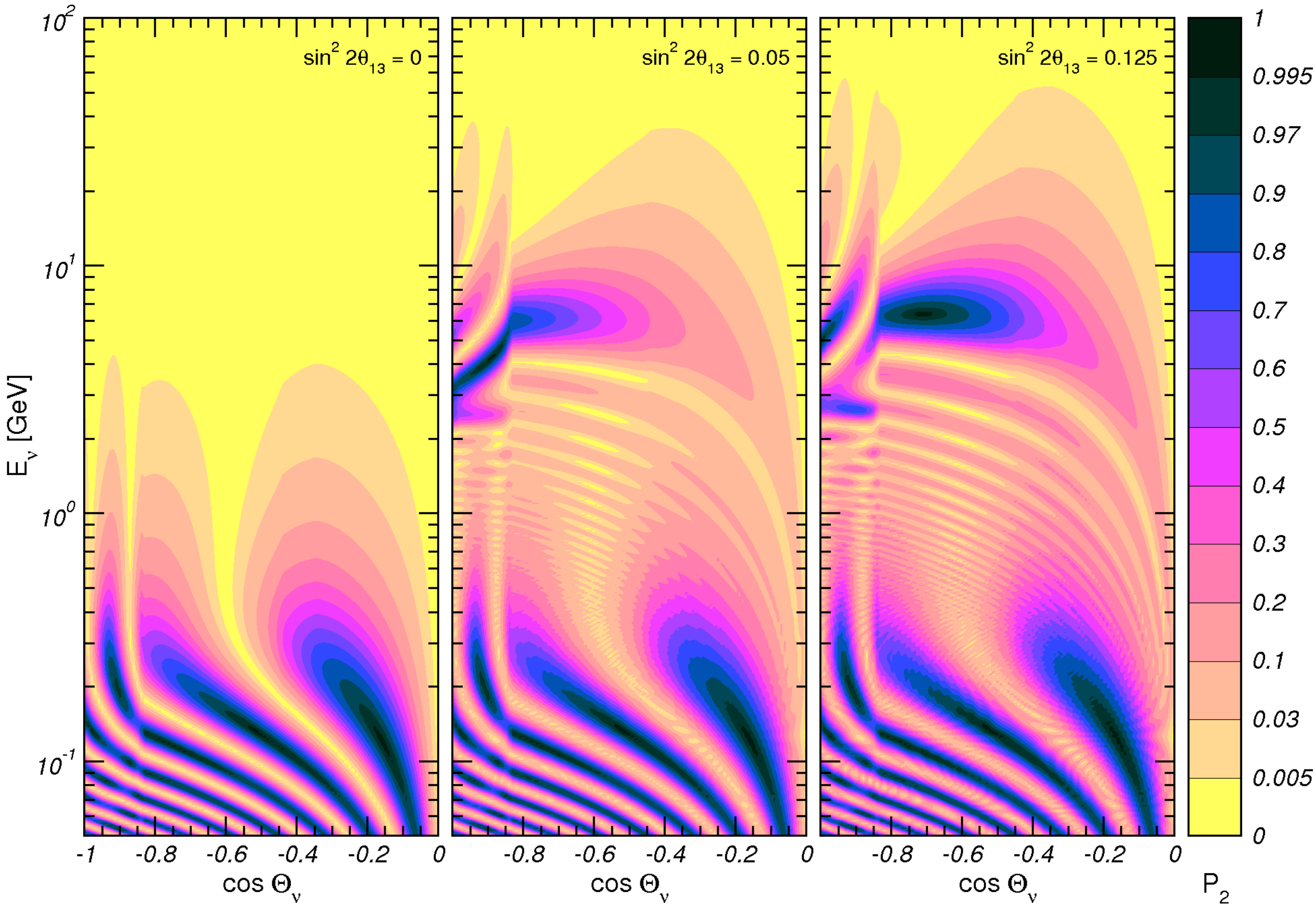
– generalized phase cond.



Other osc. channels



Including the effects of $\Delta m^2_{\rm sol}$: $(1 - P_{ee})$



Including the effects of $\Delta m^2_{\rm sol}$

Fundamental $\not{C}\not{P}$ and $\not{T}$; dependence of P_{ab} on $\delta_{\rm CP}$
(also in CP - and T - even terms) $\Rightarrow$
parameter correlations and degeneracies (e.g. θ_{13} and $\delta_{\rm CP}$)

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"Magic" baseline: $L \simeq 7300$ km – dependence on $\delta_{\rm CP}$, $\Delta m^2_{\rm sol}$ disappears (Barger *et al.*, 2001; Huber & Winter, 2003; Huber *et al.*, 2006)

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Physical interpretation (Smirnov, 2006):

$$\diamondsuit \qquad P_{e\mu} = \left|c_{23}\, A_S\, e^{i\delta_{\rm CP}} + s_{23}\, A_A\right|^2$$

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$$A_S \simeq A_S(\theta_{12},\, \Delta m^2_{21})\,, \qquad A_A \simeq A_A(\theta_{13},\, \Delta m^2_{31})$$

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For $N_e \simeq const$: $|A_S| \simeq \sin 2\theta_{12}^m \sin\phi \quad \Rightarrow \quad L_{\rm magic}: \quad \phi = \pi n$

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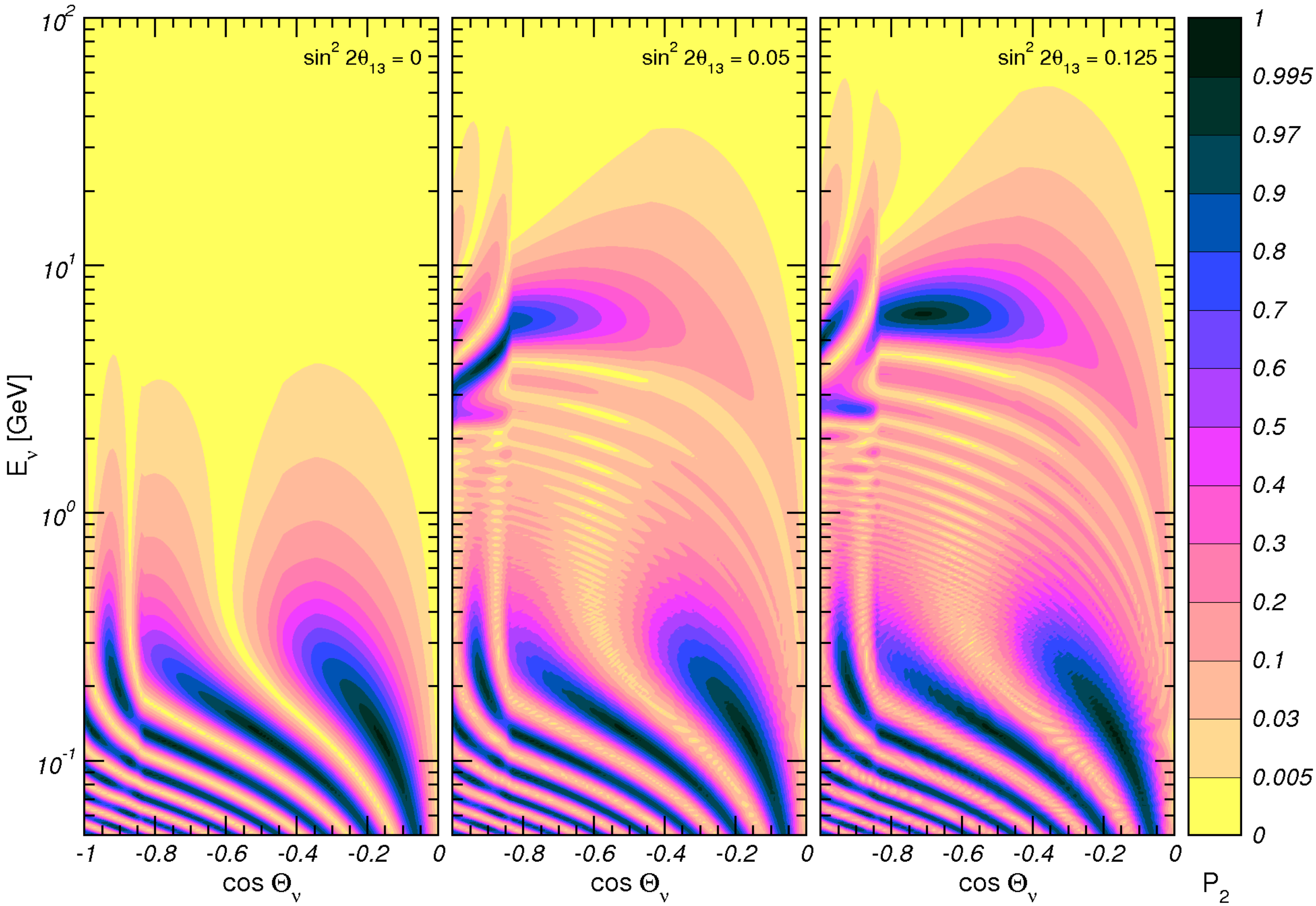
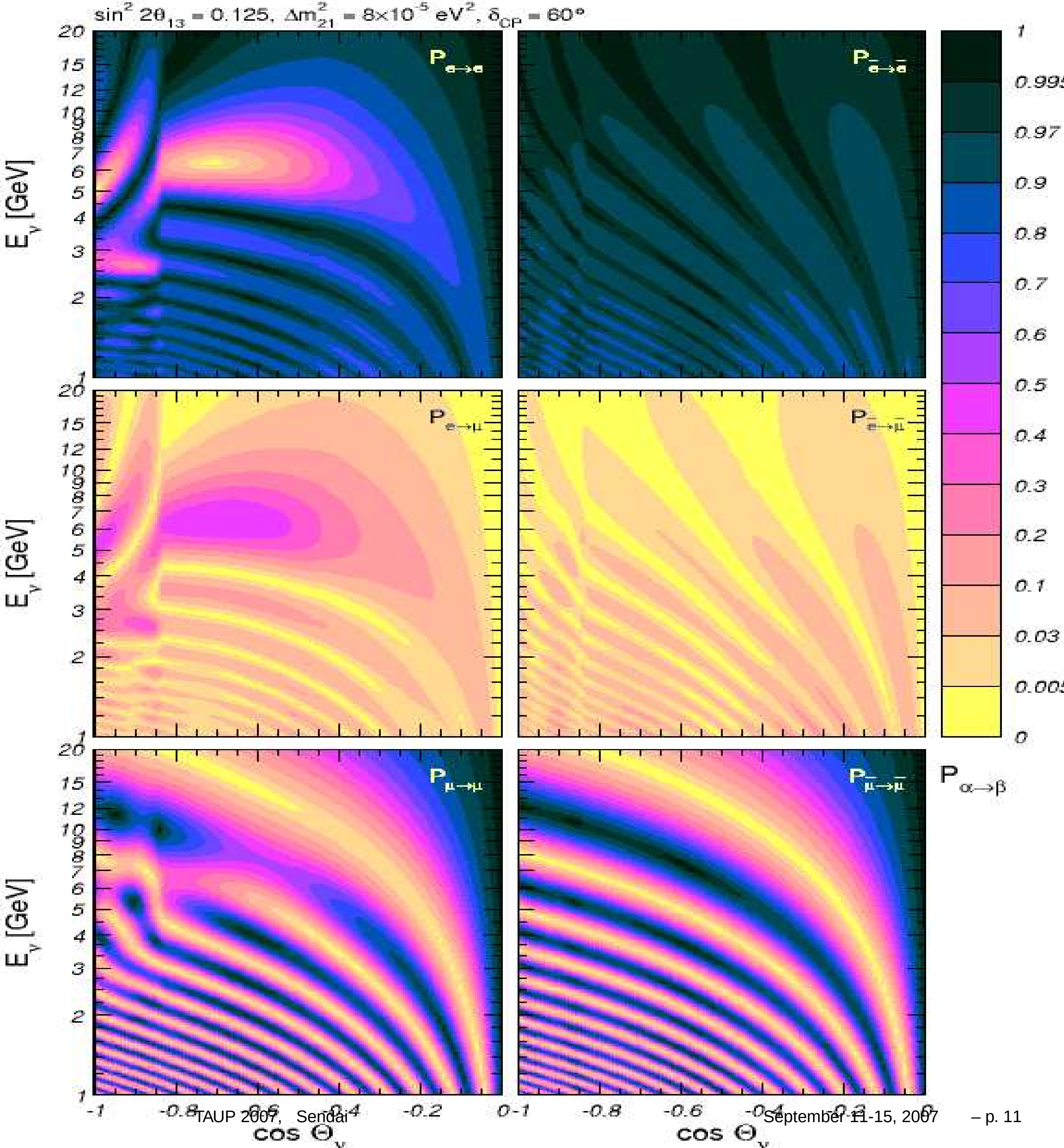


Dependence on

$\delta_{\rm CP}$



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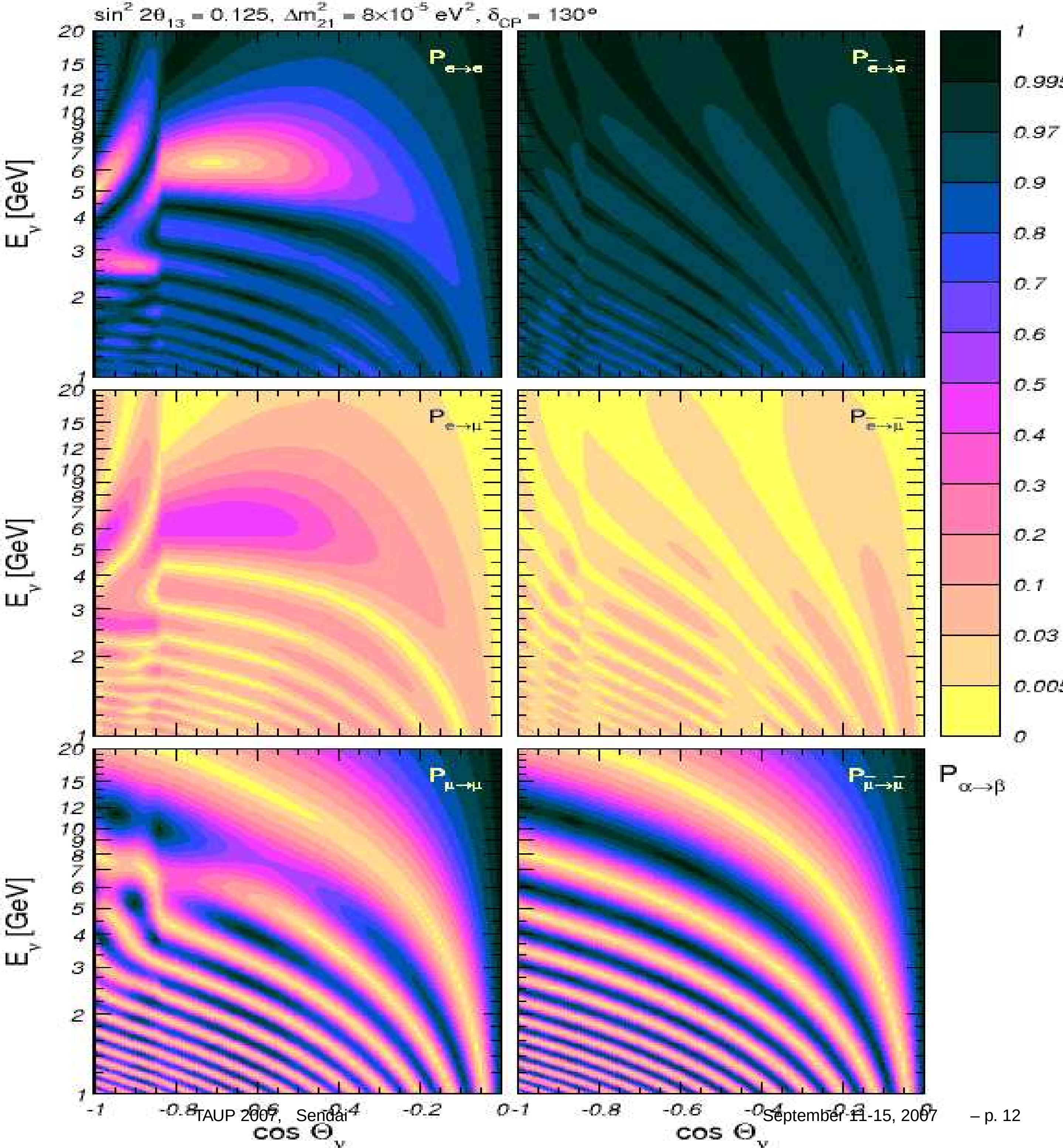

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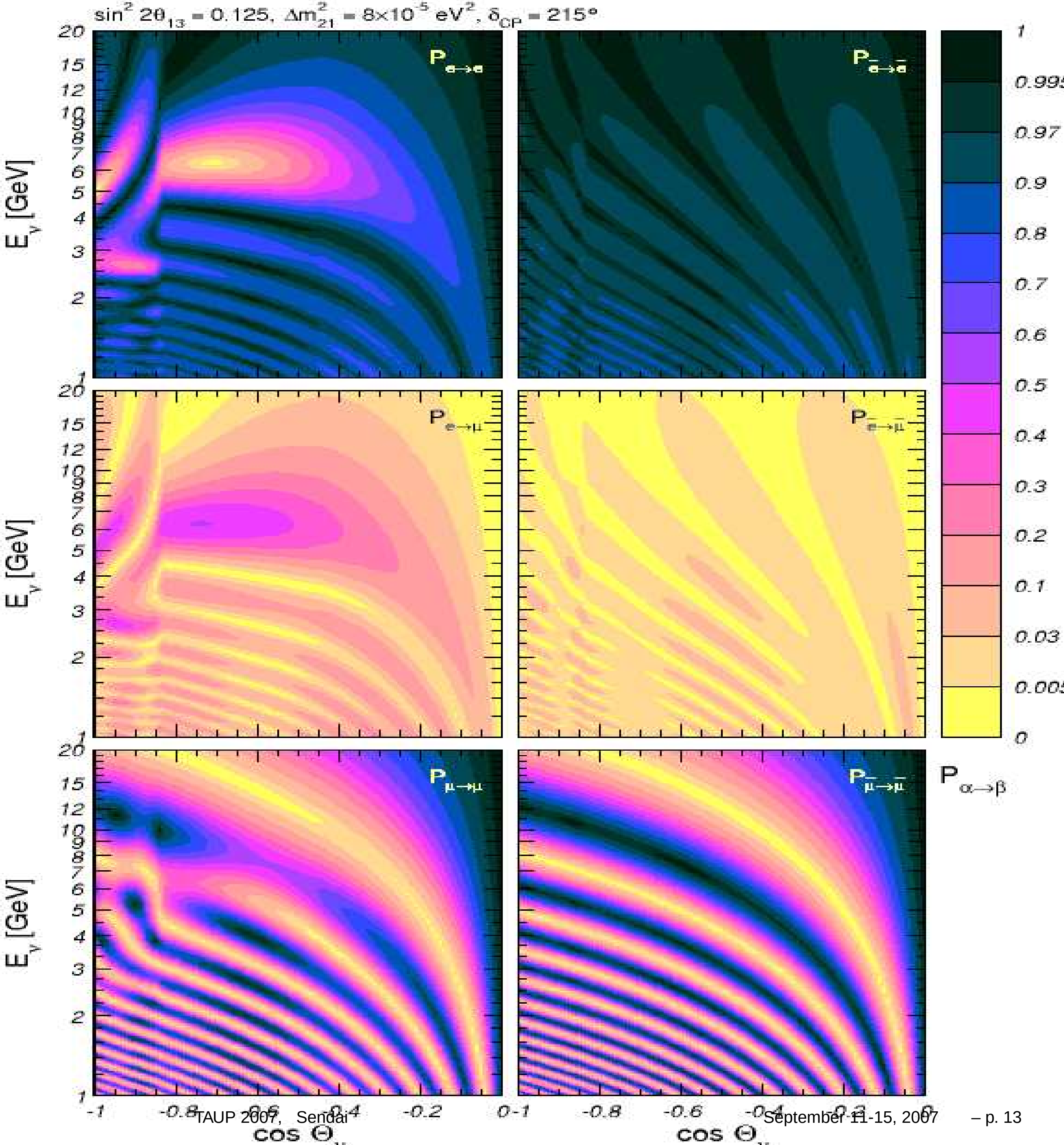

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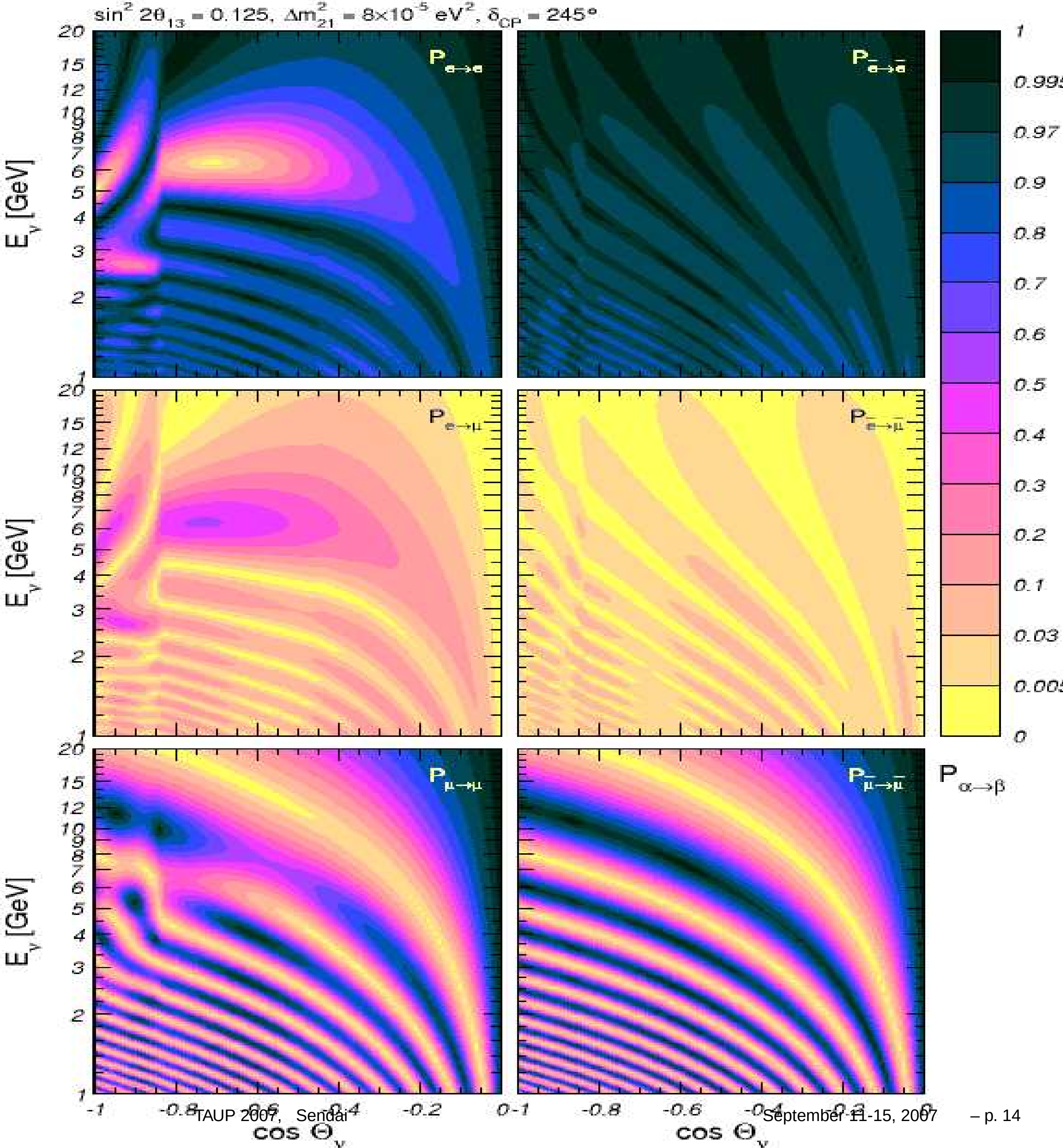

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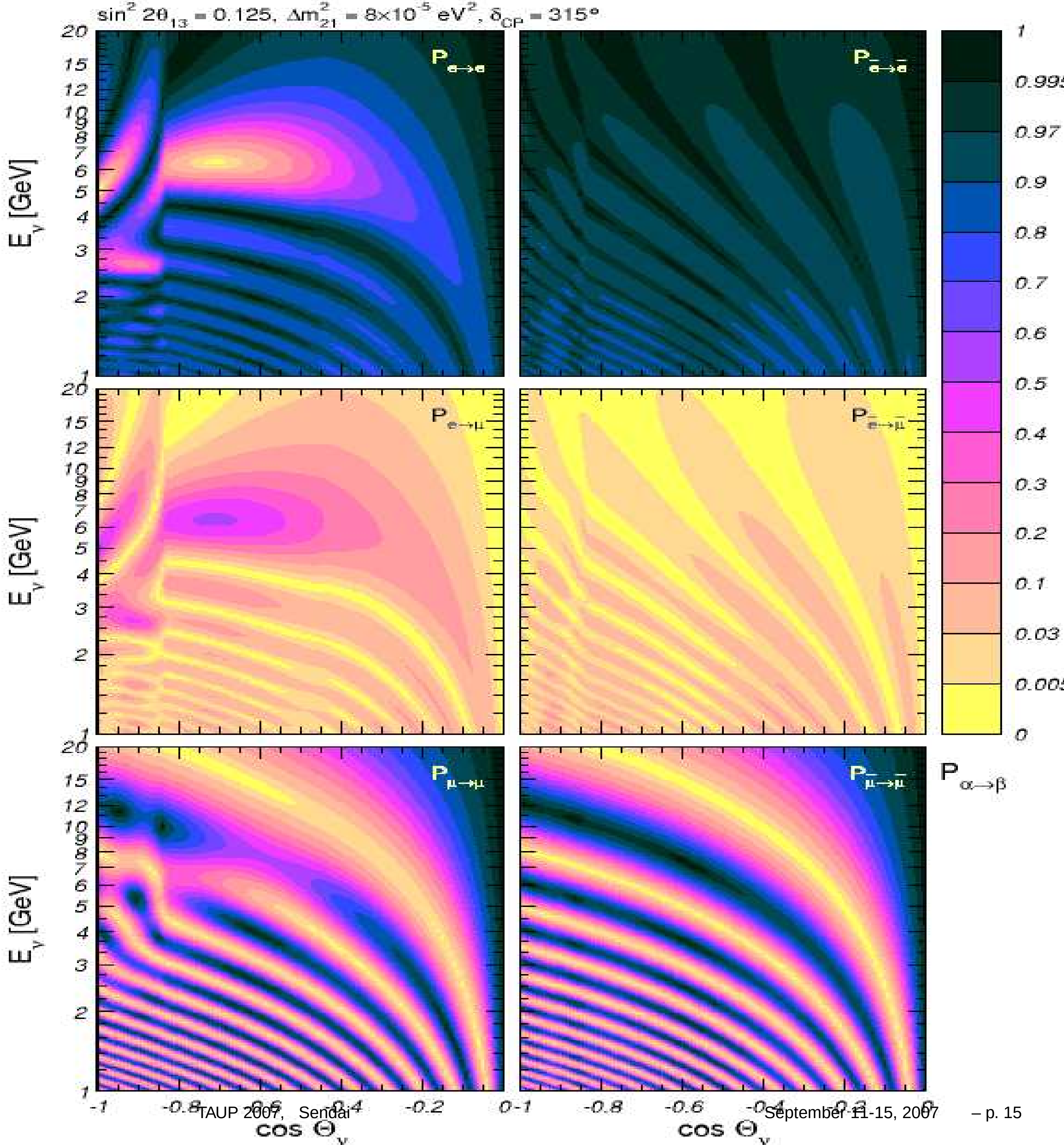

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$$\xi^2 = \frac{(E-T)^2}{E} = \frac{(\rho_{ex} - \rho_{th})^2}{\rho_{ex}} dS$$

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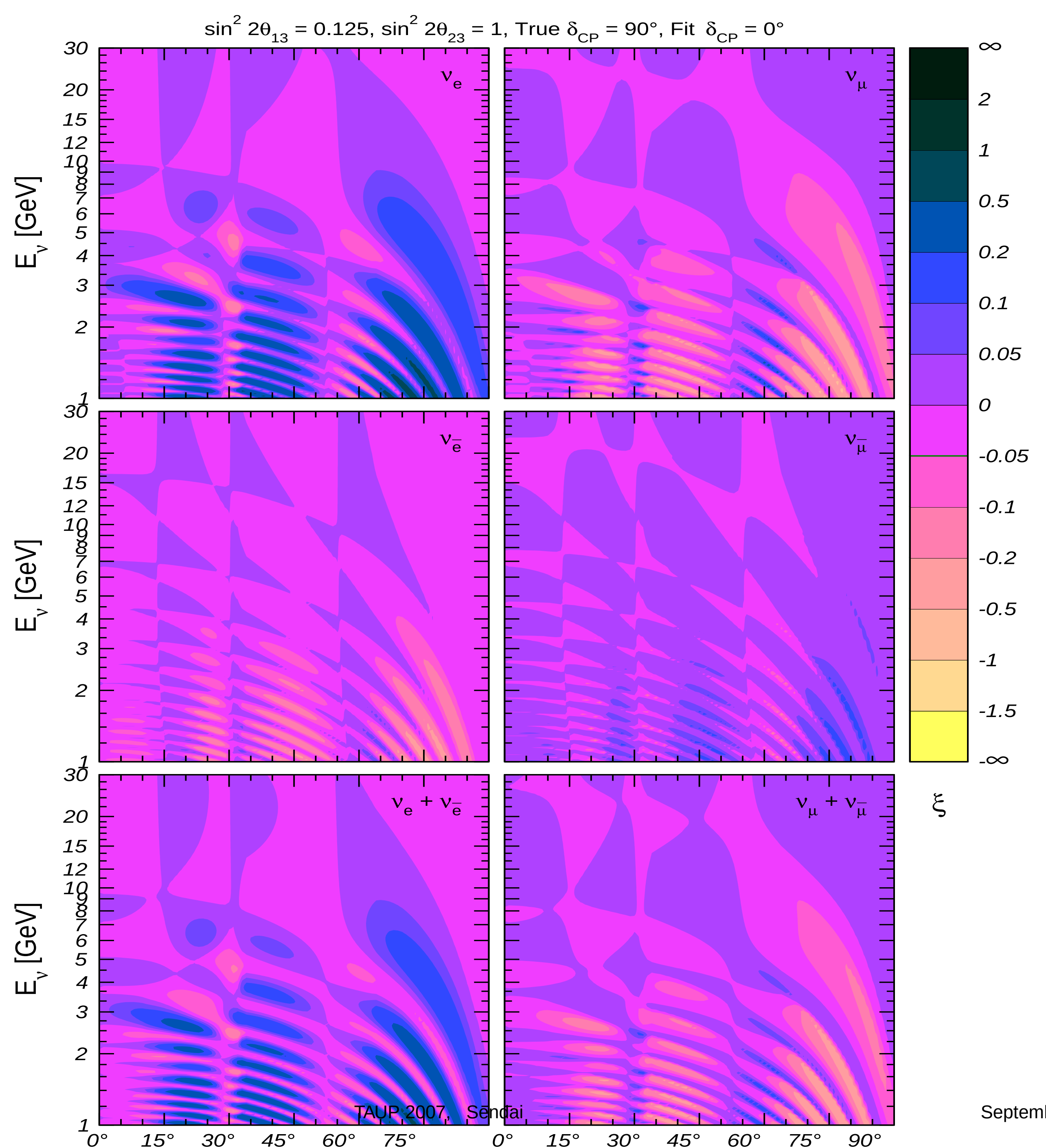
Theoretical uncertainties and systematical errors (for each particular experiment) can be incorporated. Sign of $(E-T)$ can be included.

CP oscillogram for event # difference



Understanding CP oscillograms

3 grids of curves:

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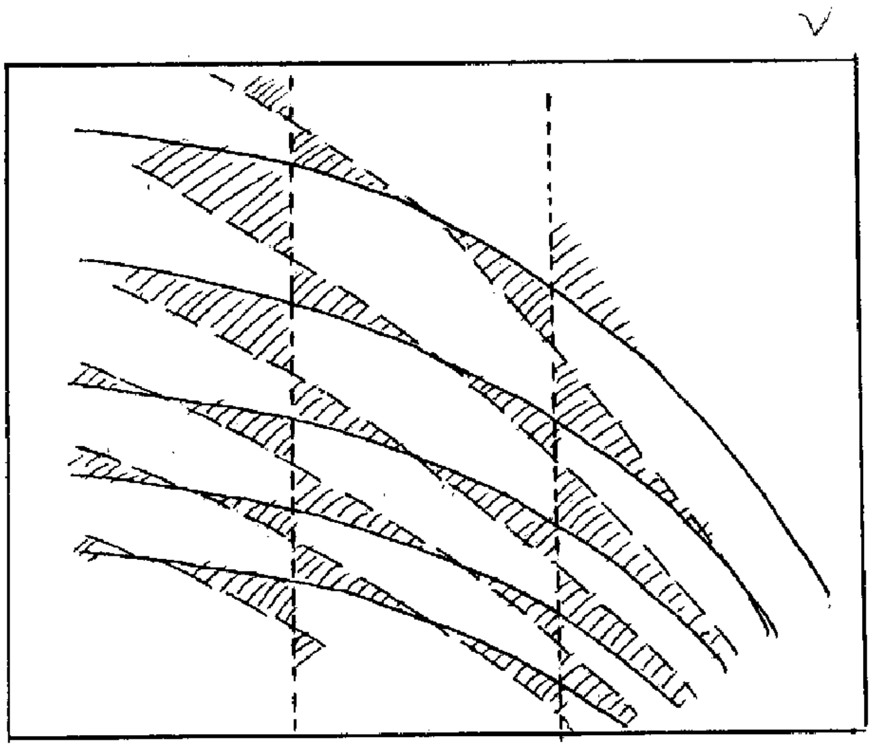
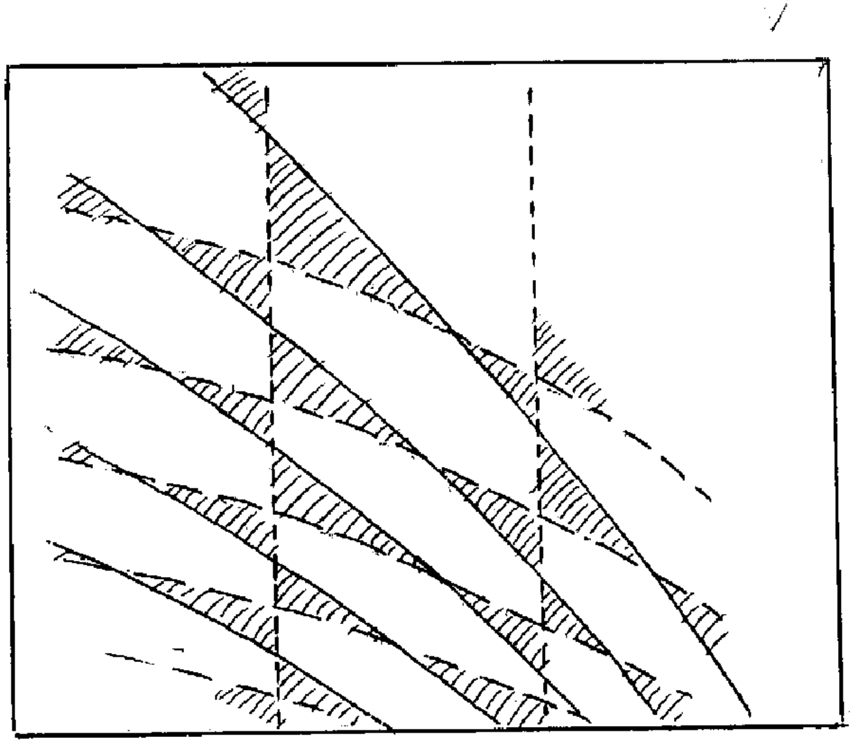
- $(\phi + \delta_{th}) = -(\phi + \delta_{true}) + 2\pi k,$ or $\phi = -(\delta_{true} + \delta_{th})/2 + \pi k.$

CP oscillograms – three grids of curves

For matter of $N_e = const.$ in the "factorization" approx.:





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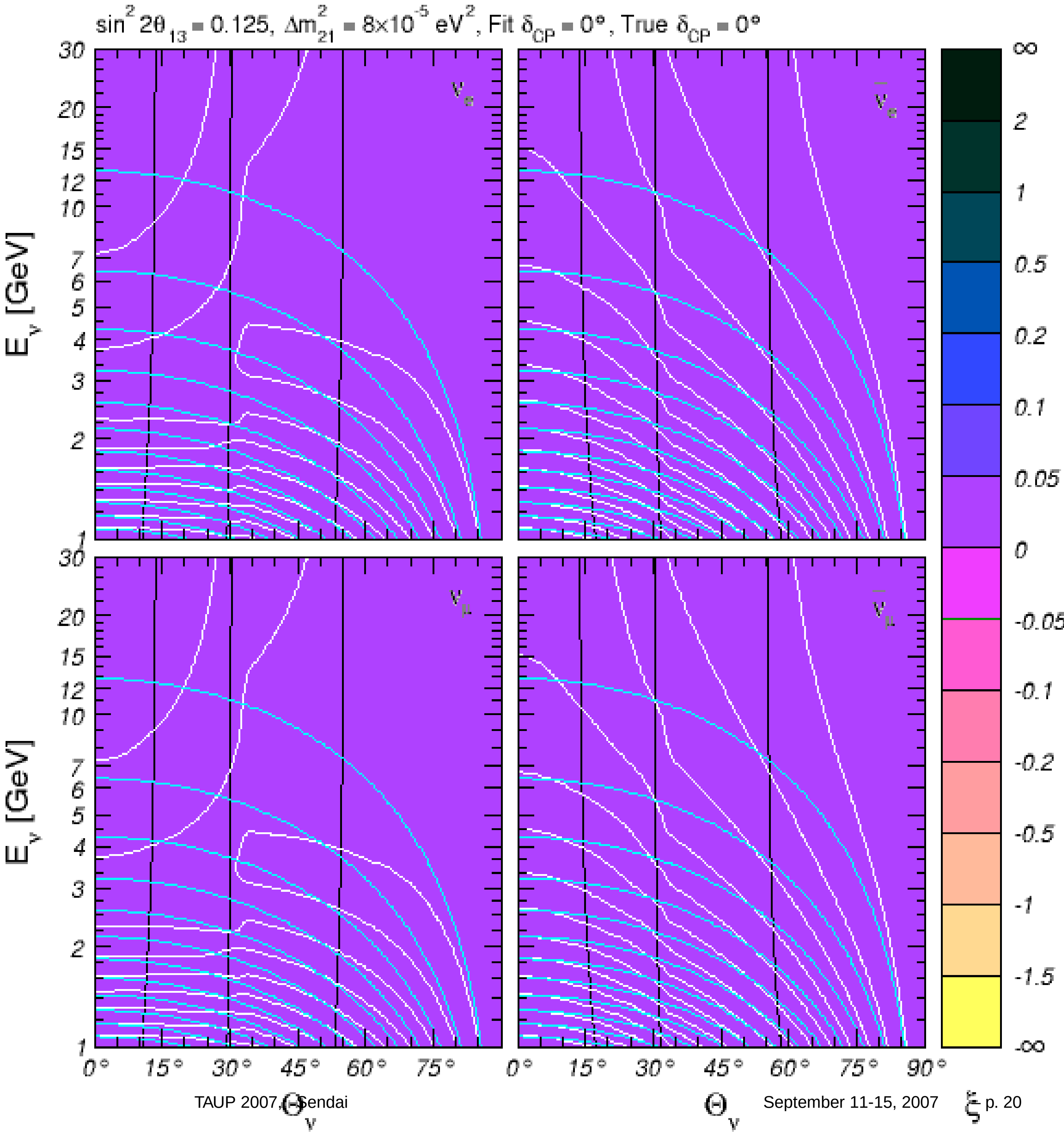

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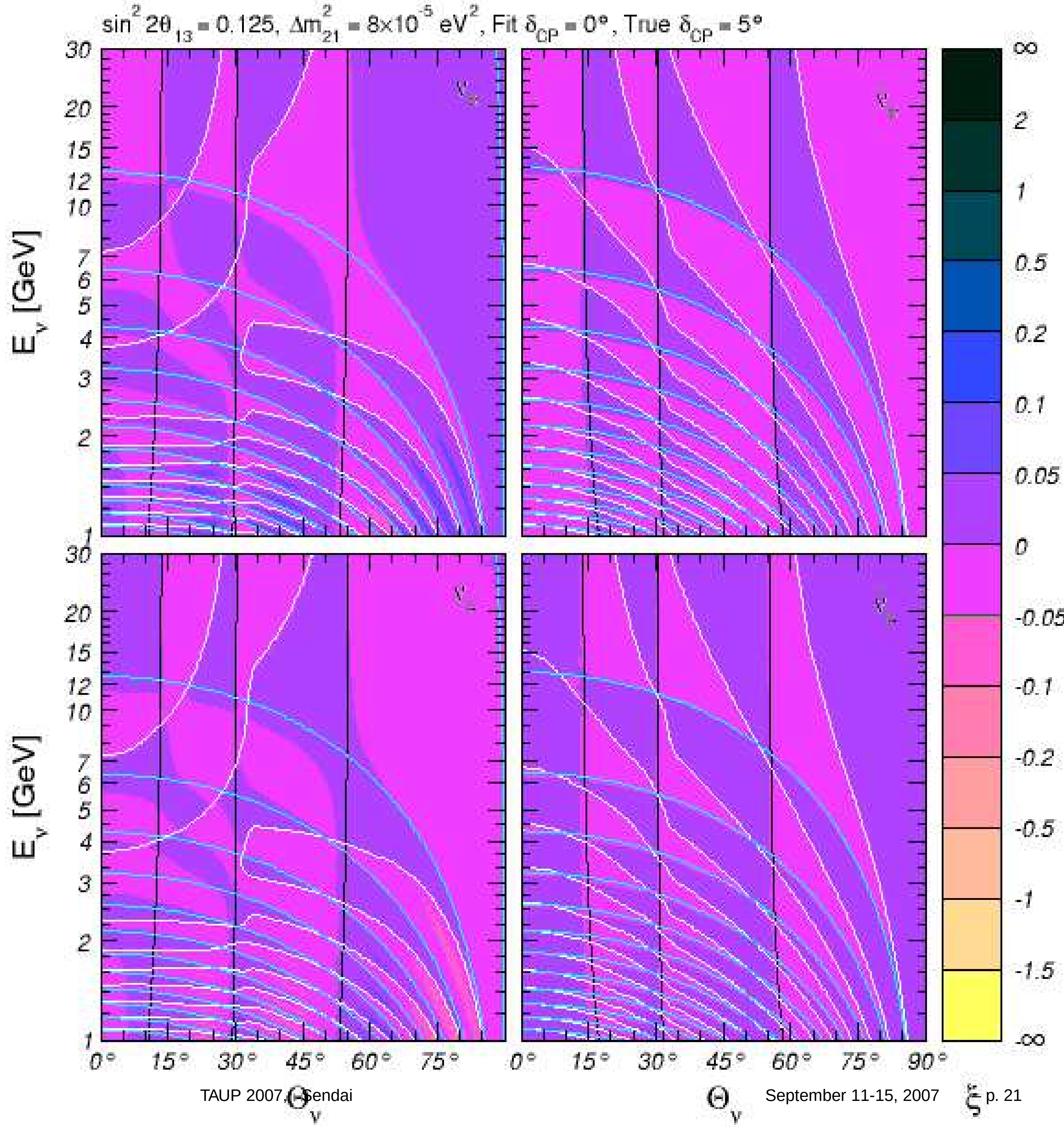

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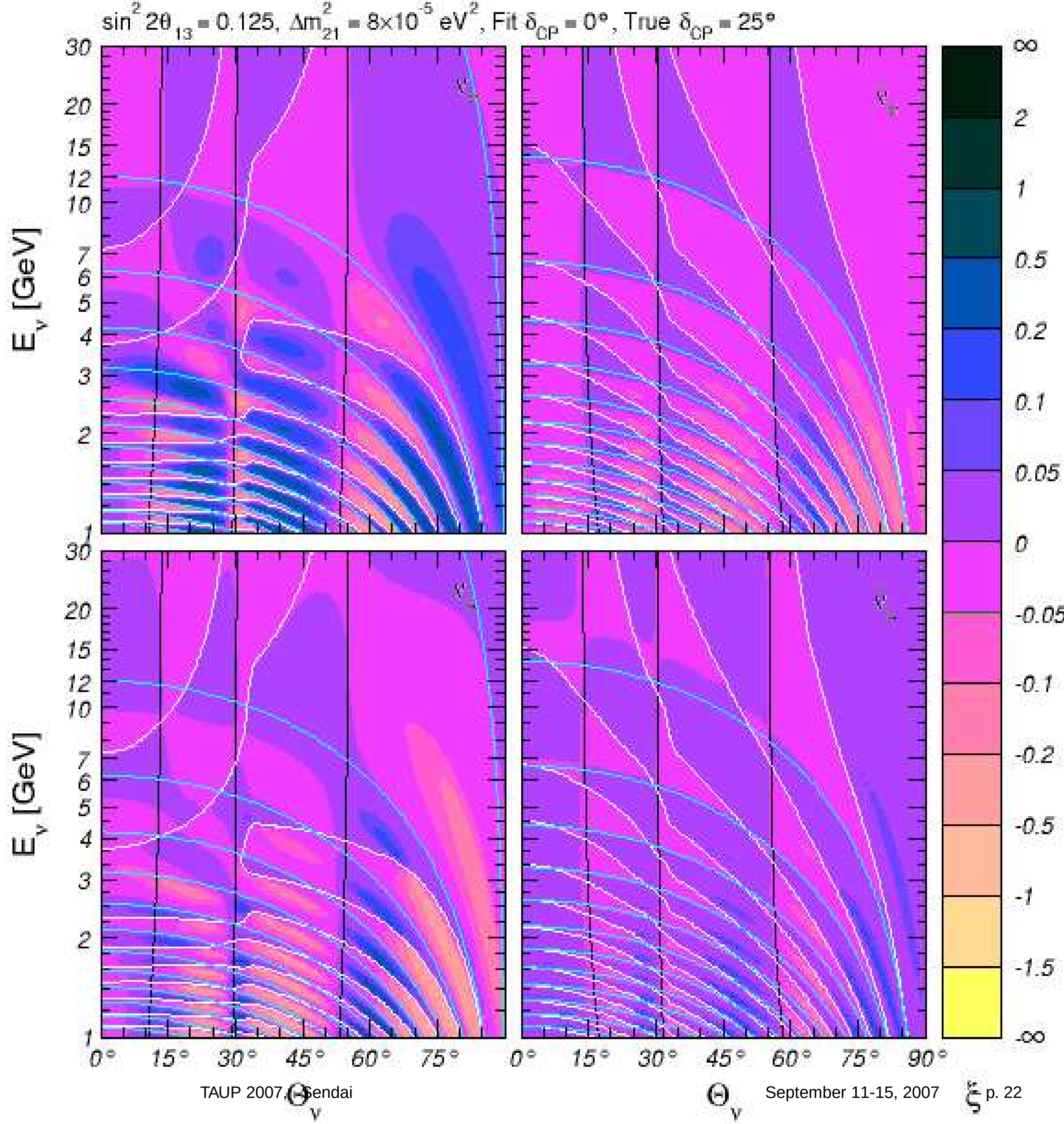

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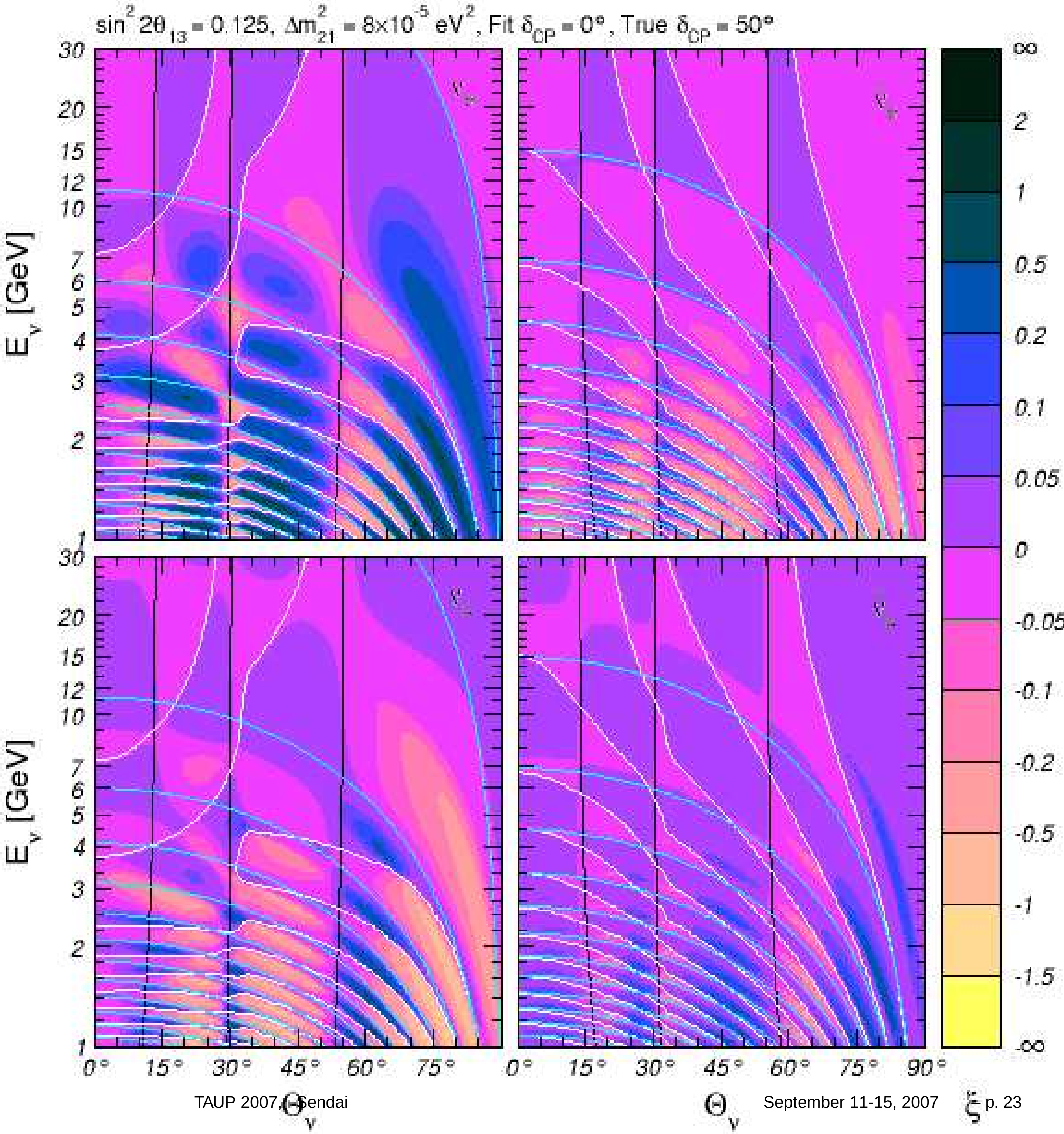

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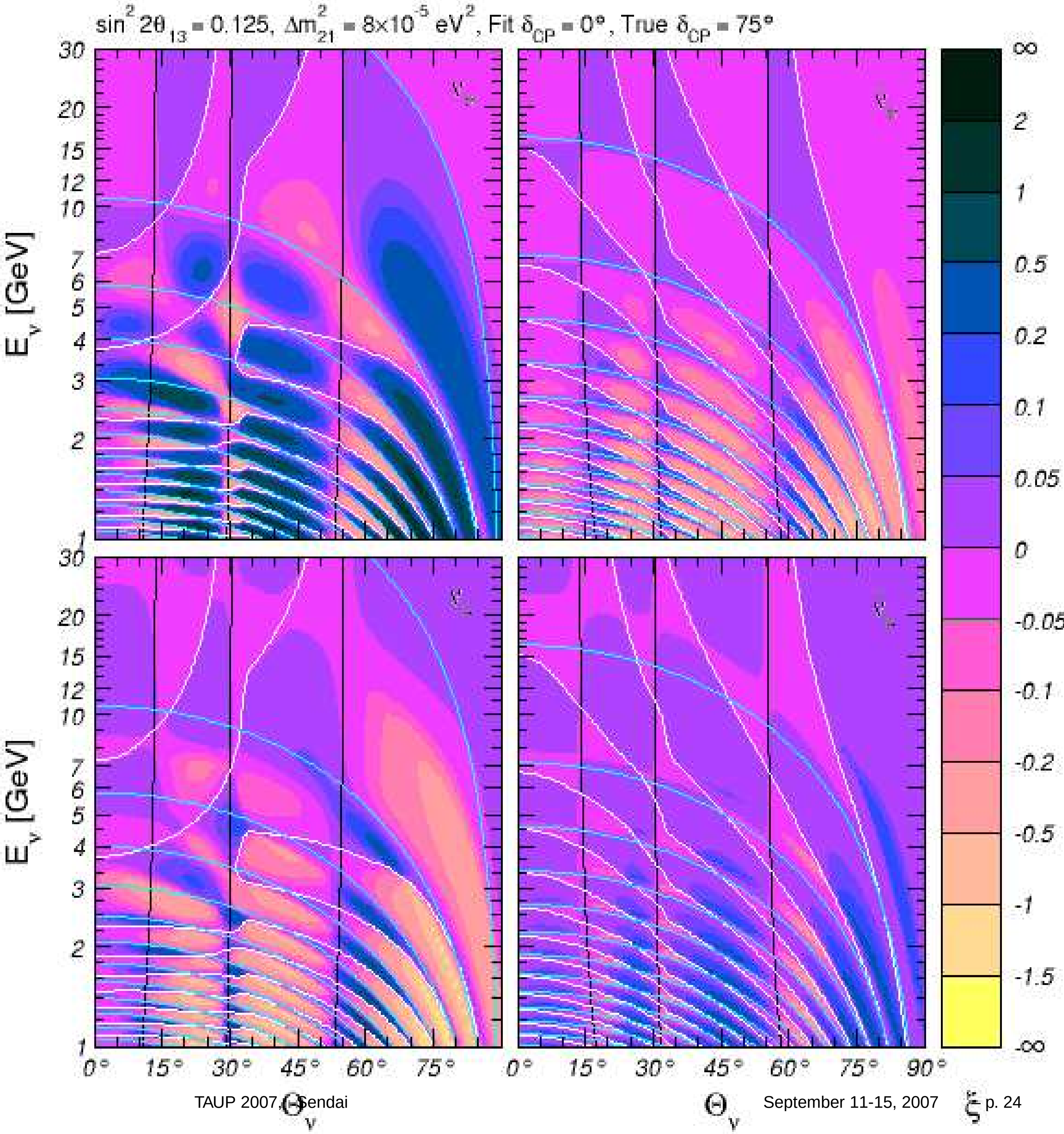

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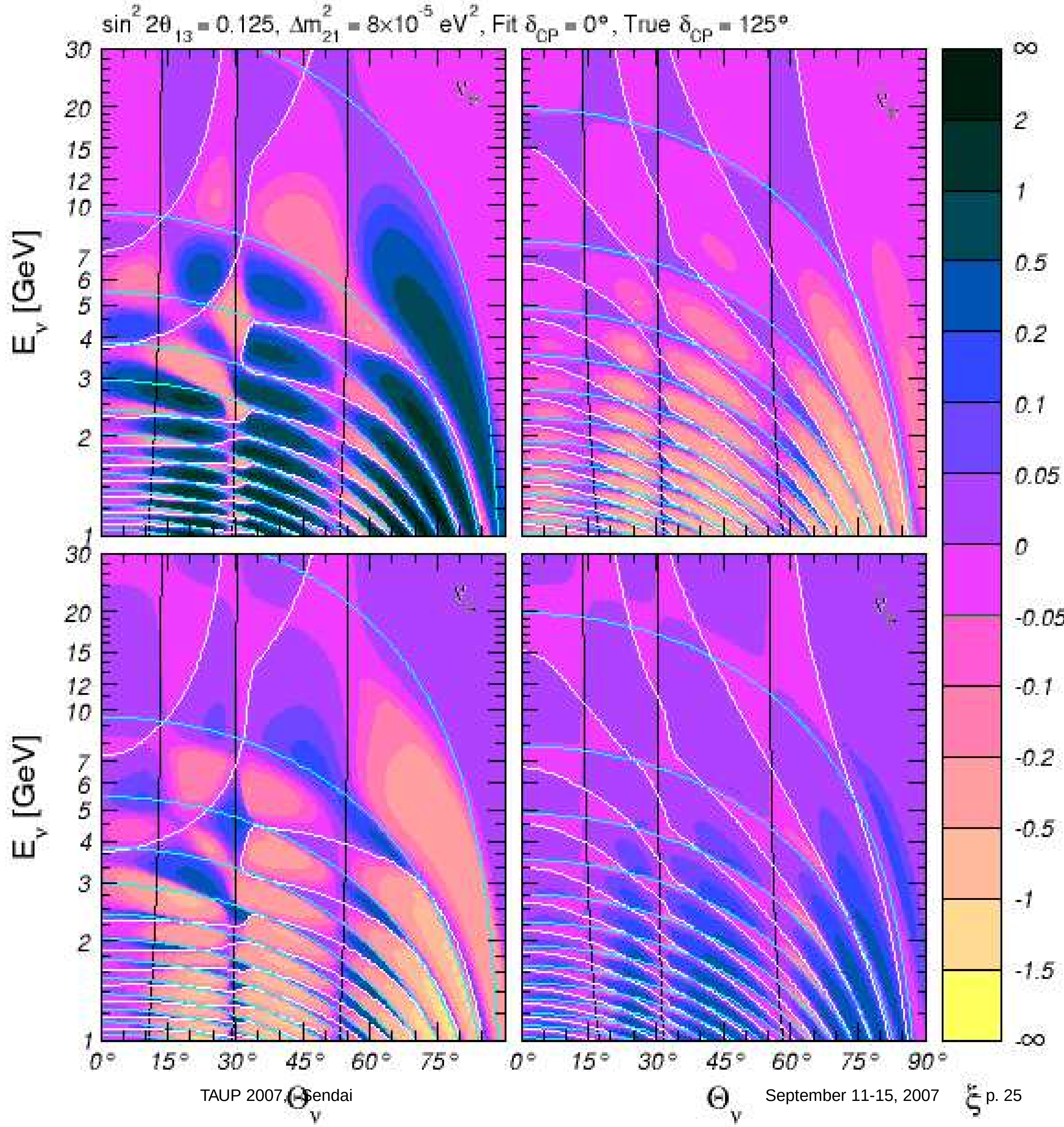

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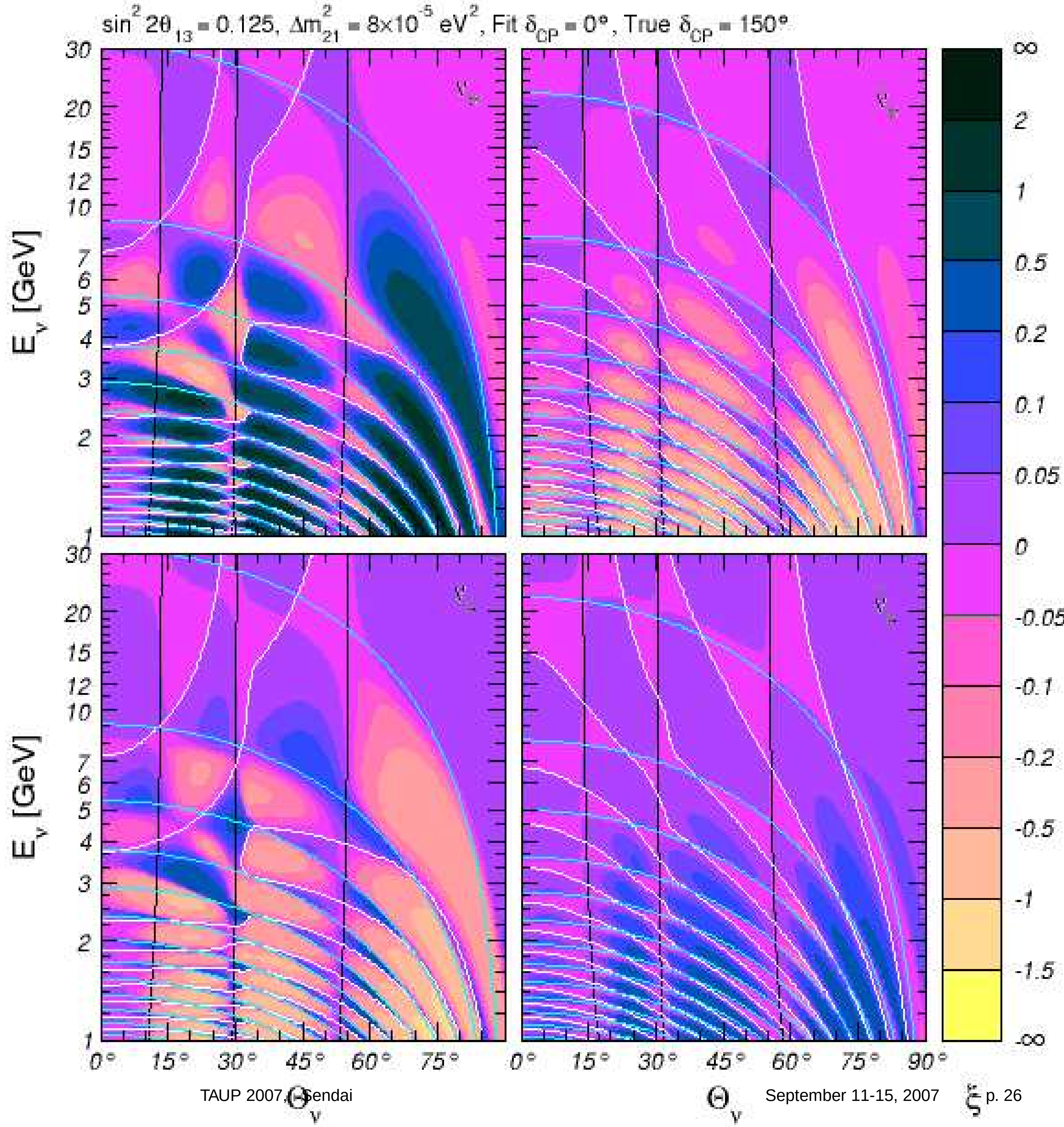

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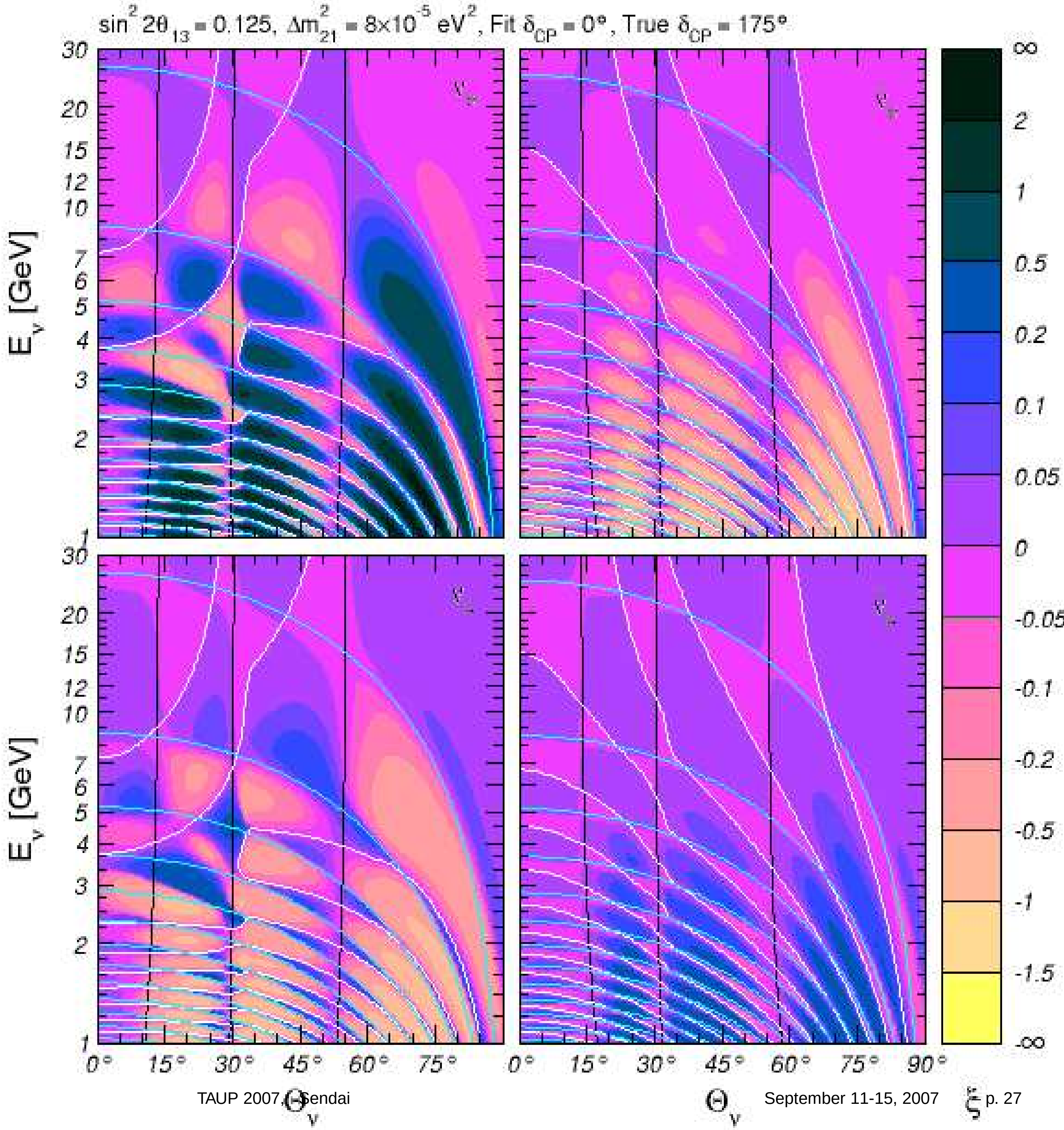

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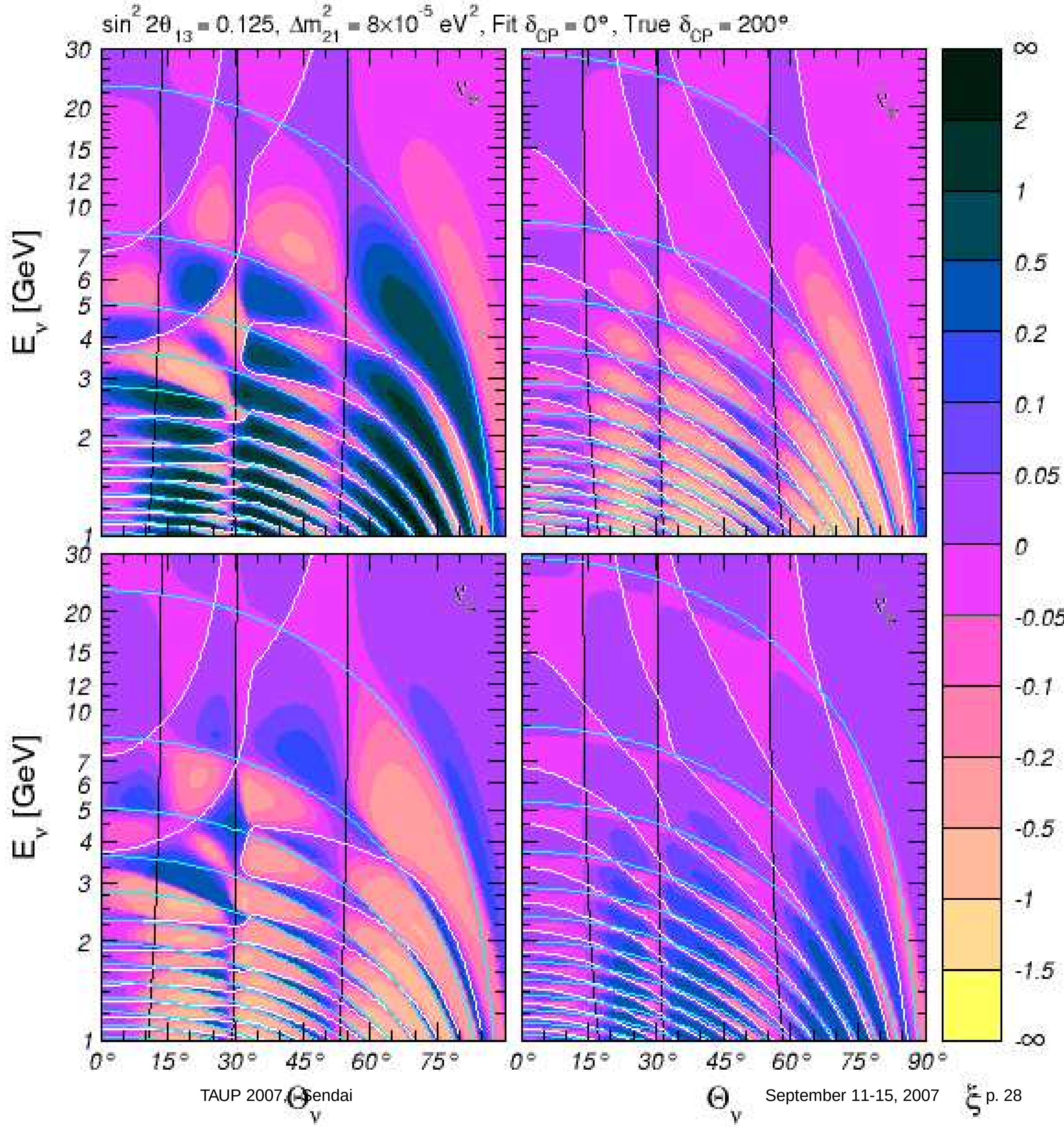

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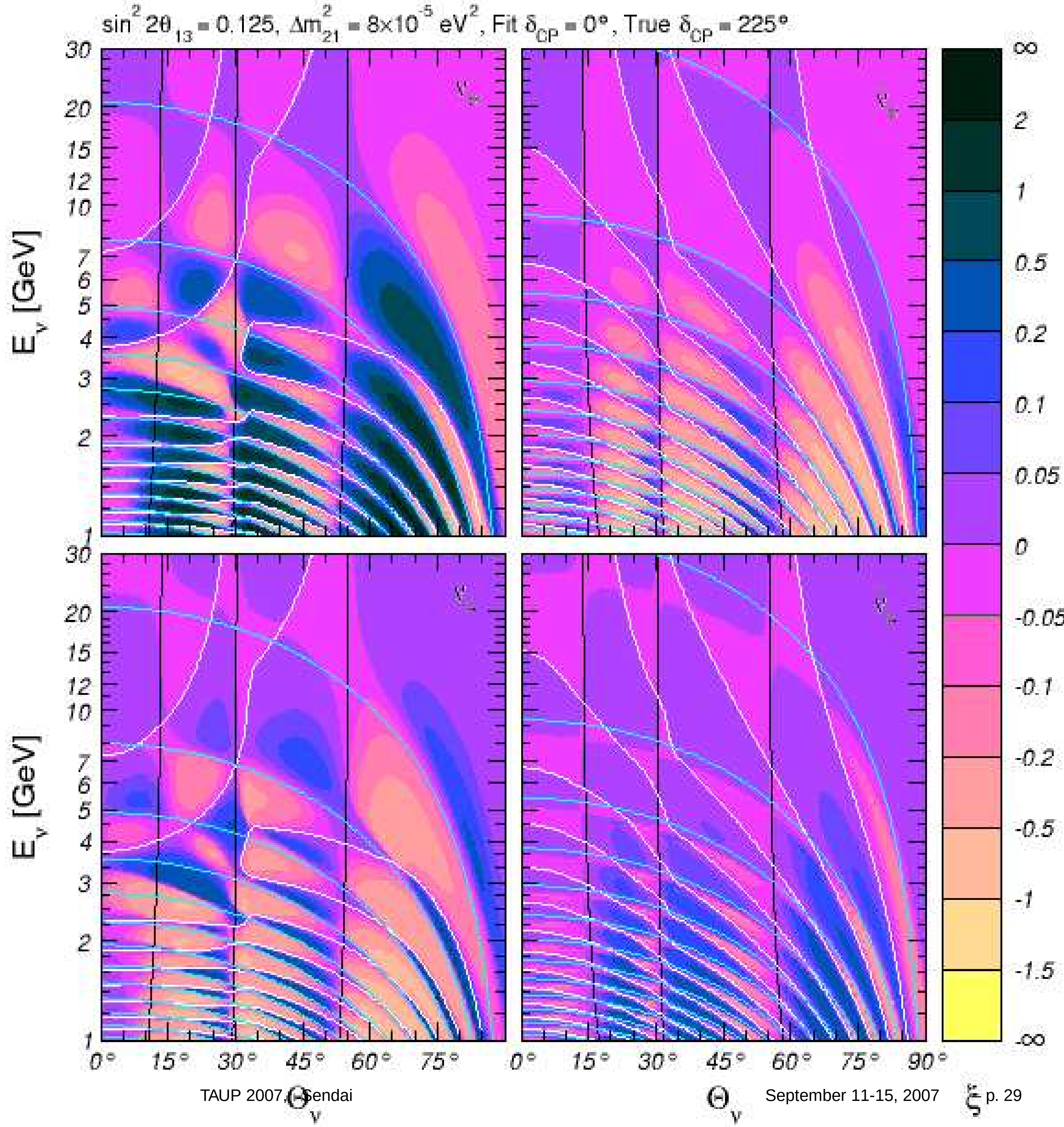

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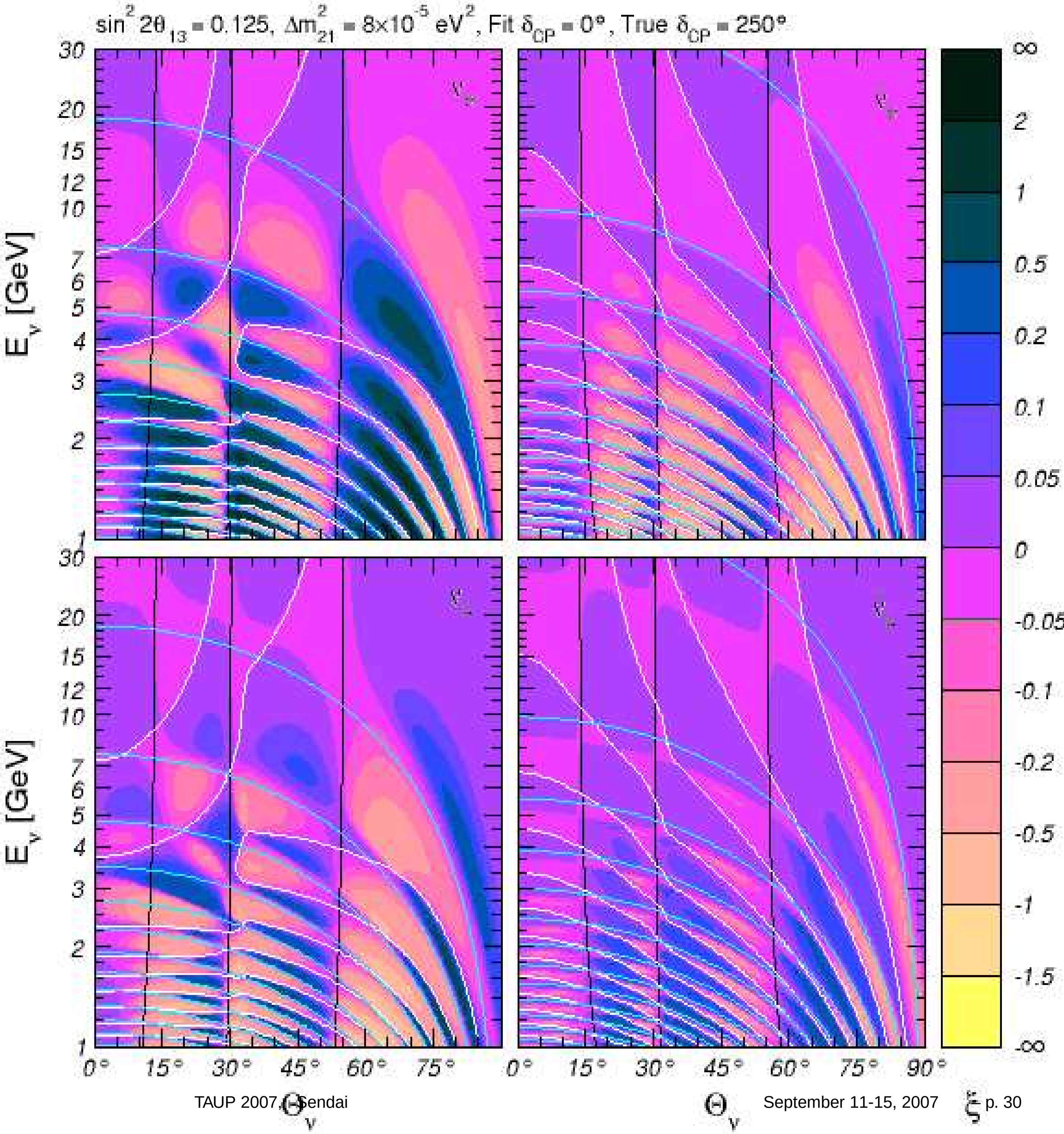
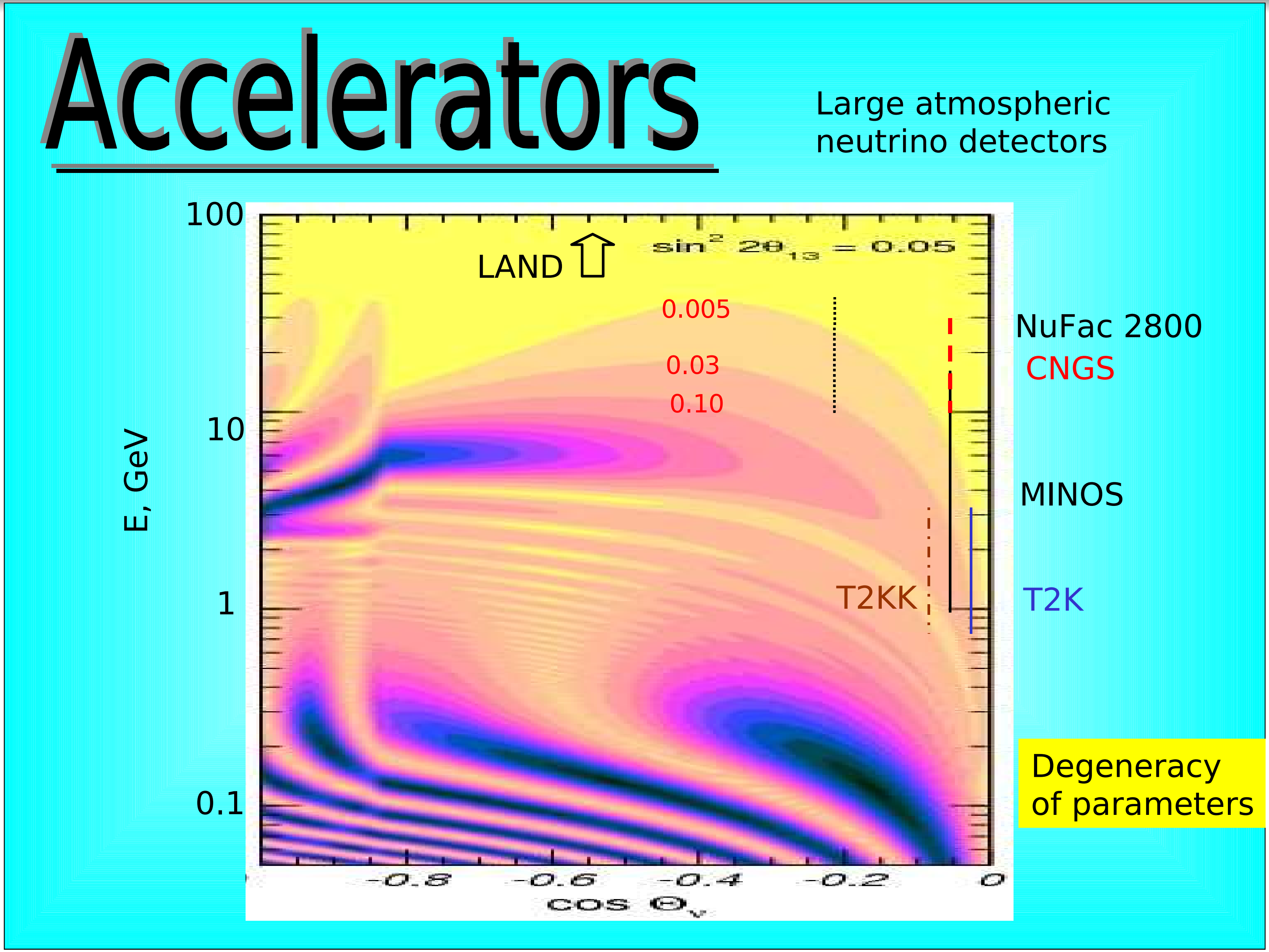


Producing the oscillograms



A. Smirnov, UCLA seminar

Studying ν properties with atm. ν's

Huge atmospheric neutrino detectors may be necessary!

Would require :

- Very good energy and angle resolution
- Low threshold ($E_{\rm thr.} \sim 3$ GeV)
- Charge discrimination (?)
- High statistics

Very ambitious, but the gain may be overwhelming $\Rightarrow$

It is worth studying the oscillograms with Huge Atmospheric Neutrino Detectors !

Conclusions

The potential of atmospheric neutrino experiments for studying neutrino properties is far from being exhausted!

Backup slides

Neutrino oscillations in the Earth

A coherent description in terms of different realizations of just 2 conditions – amplitude and phase conditions

Matter with $N_e = const$:

$$\diamondsuit \quad P_{\rm tr} \;=\; \sin^2 2\theta_m \, \sin^2 \phi_m$$

- amplitude condition = MSW resonance condition $(\theta_m = 45^\circ)$
- phase condition: $\phi_m \;=\; \pi/2 + \pi n$

Neutrino oscillations in the Earth

"Castle wall" density profile:

$$\diamondsuit \qquad P_{\rm tr}^{(n)} = \frac{X_1^2 + X_2^2}{X_1^2 + X_2^2 + X_3^2} \sin^2 n\Phi$$

Evolution matrix: $\nu(t) = U(t, t_0)\,\nu(0)$. For 2 layers:

$$U^{(2)}(t, t_0) = \begin{pmatrix} Y - iX_3 & -i(X_1 - iX_2) \\ -i(X_1 + iX_2) & Y + iX_3 \end{pmatrix}, \qquad Y^2 + \mathbf{X}^2 = 1$$

- amplitude condition = parametric resonance condition ($X_3 = 0$)
- phase condition: $\Phi \equiv \arccos Y = \pi/2 + \pi n$

The meaning of the amplitude condition

Alignment of the transitions amplitudes in different layers.

Evolution matrices for individual layers:

$$U_i(t,\,t_0) \;=\; \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i^* & \alpha_i^* \end{pmatrix}, \qquad |\alpha_i|^2 + |\beta_i|^2 \;=\; 1\,, \qquad i = 1,2,3$$

For 2 layers: $U^{(2)} = U_2 U_1$,

$$\beta^{(2)} \;=\; \alpha_2\,\beta_1 \;+\; \beta_2\,\alpha_1^*$$

Alignment (collinearity) condition:

$$\arg(\alpha_2\,\beta_1) \;=\; \arg(\beta_2\,\alpha_1^*) \qquad \mathrm{mod}\ (\pi)$$

– potentially leads to maximal trans. probability.

For 2 layers of const. densities: align. cond. $\Leftrightarrow$ $s_1 s_2 X_3 \;=\; 0$

How about 3 layers?

$U^{(3)} = U_3 U_2 U_1$. For the Earth, $U^{(3)} = U_1^T U_2 U_1$.

Transition amplitude:

$$\beta^{(3)} = \alpha_1 \alpha_2 \beta_1 - \alpha_1^* \alpha_2^* \beta_1^* + |\alpha_1|^2 \beta_2 + |\beta_1|^2 \beta_2^*$$

$\Rightarrow$ If the 2-layer align. cond. is satisfied, so is the 3-layer one !

A consequence of

- The symmetry of the core density profile
- The symmetry of the overall density profile of the Earth (3rd layer's profile is the reverse of the 1st layer's one)

$\Rightarrow$ The generalized amplitude condition is the alignment condition in the case of non-constant density layers

Generalized phase condition

For constant density matter: $\phi \;=\; \pi/2 + \pi n \quad \Leftrightarrow \quad \mathrm{Im}\, \alpha^{(1)} \beta^{(1)*} = 0$.

$\Rightarrow$ Generalize to an arbitrary density profile:

$$\mathrm{Im}\, \alpha \beta^* \;=\; 0 \qquad \Leftrightarrow \qquad \frac{dP_{\rm tr}}{dL} \;=\; 0$$

The whole complex oscillation pattern:

- MSW resonances
- parametric resonances
- saddle points
- local maxima and minima
- absolute maxima and minima

can be understood in terms of the generalized amplitude and phase conditions ! (E.A., Maltoni & Smirnov, 2006)

Special points

- $\mathrm{Re}\,\alpha_1 \;=\; \mathrm{Re}\,\alpha_2 \;=\; 0$ (constant-density layers: $c_1 \;=\; c_2 \;=\; 0$)

$$\Rightarrow \qquad P_A \;=\; \sin^2(4\theta_m - 2\theta_c)$$

Maxima between the core and mantle MSW resonances for $\theta_c - \theta_m > \pi/4$ and above the MSW resonances, and saddle points below the MSW resonances and between the resonances for $\theta_c - \theta_m \leq \pi/4$.

- $\mathrm{Im}\,\beta_1 \;=\; \mathrm{Re}\,\alpha_2 \;=\; 0$ (constant-density layers: $s_1 \;=\; c_2 \;=\; 0$)

$$\Rightarrow \qquad P_A \;=\; \sin^2 2\theta_c$$

Lie below $2.5\ \mathrm{MeV}$. Local maxima. No (or almost no) oscillation effect in the mantle.

Three grids of curves ($N_e = const.$, fact. appr.)

(1) Solar "magic":

$$VL = 2\pi n$$

There are only 3 such lines with $n = 1, 2$ and 3.

(2) Atm. "magic curves" $\omega_{13} L = \pi k$. For E not too close to $(E_{atm})_{res}$, $\omega_{13} \simeq (\Delta \mp V)/2$ for neutrinos (antineutrinos) ($\Delta \equiv \Delta m_{31}^2/(2E)$) $\Rightarrow$

$$E = \frac{\Delta m_{31}^2 L}{4\pi k \pm 2VL}\,.$$

(3) The interference phase condition $\phi = -(\delta_{true} + \delta_{th})/2 + \pi n_1\,.$
$\phi = (\Delta m_{31}^2/4E)L \quad \Rightarrow$

$$\frac{\Delta m_{31}^2}{4E} L = -\frac{\delta_{true} + \delta_{th}}{2} + \pi n_1\,,$$

or

$$E = \frac{\Delta m_{31}^2 L}{4C_0 + 4\pi n_1}\,, \qquad C_0 = -\frac{\delta_{true} + \delta_{th}}{2}\,.$$

General dependence on $\delta_{\rm CP}$

Rotate by

$$O'_{23} = O_{23} \times {\rm diag}(1, 1, e^{i\delta_{\rm CP}})$$

From commutativity of ${\rm diag}(V(t), 0, 0)$ with O'_{23} $\Rightarrow$
General dependence of probabilities on $\delta_{\rm CP}$:

$$P_{e\mu} = A_{e\mu} \cos\delta_{\rm CP} + B_{e\mu} \sin\delta_{\rm CP} + C_{e\mu}$$

$$P_{\mu\tau} = A_{\mu\tau} \cos\delta_{\rm CP} + B_{\mu\tau} \sin\delta_{\rm CP} + C_{\mu\tau}$$

$$+ D_{\mu\tau} \cos 2\delta_{\rm CP} + E_{\mu\tau} \sin 2\delta_{\rm CP}$$

(Yokomakura, Kimura & Takamura, 2002)

Evolution in the rotated basis

Evolution matrix $S(t,\, t_0)$: $\nu(t) \;=\; S(t,\, t_0)\, \nu(t_0)$. Satisfies

$$\diamondsuit \quad i\frac{d}{dt}S(t,\, t_0) \;=\; H\, S(t,\, t_0) \qquad \text{with} \quad S(t_0,\, t_0) \;=\; \mathbb{1}\,.$$

$$H \;=\; (O_{23}\, \Gamma_\delta\, O_{13}\, \Gamma_\delta^\dagger\, O_{12}\,)\, \mathrm{diag}(0,\, \delta,\, \Delta)\; (O_{12}^T\, \Gamma_\delta O_{13}^T\, \Gamma_\delta^\dagger\, O_{23}^T) \;+\; \mathrm{diag}(V(t),\, 0,\, 0)$$

$$=\; (O_{23}\, \Gamma_\delta\, O_{13}\;\, O_{12}\,)\, \mathrm{diag}(0,\, \delta,\, \Delta)\, (O_{12}^T\, O_{13}^T\, \Gamma_\delta^\dagger\, O_{23}^T) \;+\; \mathrm{diag}(V(t),\, 0,\, 0)$$

where

$$\delta \;\equiv\; \frac{\Delta m_{21}^2}{2E}\,, \qquad \Delta \;\equiv\; \frac{\Delta m_{31}^2}{2E}$$

Oscillation probabilities:

$$P_{ab} \;=\; |S_{ba}|^2$$

Define

$$O_{23}' = O_{23}\, \Gamma_\delta$$

The matrix $\mathrm{diag}(V(t), 0, 0)$ commutes with O_{23}' $\Rightarrow$ go to the rotated basis

Evolution in the rotated basis – contd.

$$\nu \;=\; O'_{23}\,\nu'\,, \qquad \text{or} \qquad S(t,\,t_0) \;=\; O'_{23}\,S'(t,\,t_0)\,O'_{23}{}^{\dagger}\,,$$

In the rotated basis $H' \;=\; O'_{23}\,H\,O'_{23}{}^{\dagger}$. Explicitly:

$$H'(t) = \begin{pmatrix} s_{12}^2 c_{13}^2 \delta + s_{13}^2 \Delta + V(t) & s_{12} c_{12} c_{13} \delta & s_{13} c_{13} \left(\Delta - s_{12}^2 \delta\right) \\ s_{12} c_{12} c_{13} \delta & c_{12}^2 \delta & -s_{12} c_{12} s_{13} \delta \\ s_{13} c_{13} \left(\Delta - s_{12}^2 \delta\right) & -s_{12} c_{12} s_{13} \delta & c_{13}^2 \Delta + s_{12}^2 s_{13}^2 \delta \end{pmatrix}$$

Dependence on θ_{23} and $\delta_{\rm CP}$ can be obtained in the general case by rotating back to the original flavour basis. Also: easy to apply PT approximations

- If $\frac{\Delta m_{21}^2}{2E} L \ll 1$ – neglect $\delta \;=\; \frac{\Delta m_{21}^2}{2E}$
- If θ_{13} is very small – neglect s_{13}

or use expansion in these small parameters