Gravitational waves from Extreme mass ratio inspirals

Gravitational Radiation Reaction Problem

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Various sources of gravitational waves

• **Inspiraling binaries**
• (Semi-) periodic sources
  – Binaries with large separation (long before coalescence)
    • a large catalogue for binaries with various mass parameters with distance information
  – Pulsars
• **Sources correlated with optical counter part**
  – supernovae
  – X-ray burst
• **Stochastic background**
  – GWs from the early universe
  – Unresolved foreground
Inspiraling binaries

In general, binary inspirals bring information about
- Event rate
- Binary parameters
- Test of GR

• Stellar mass BH/NS
  - Target of ground based detectors
  - NS equation of state
  - Possible correlation with short \( \gamma \)-ray burst
  - Primordial BH binaries (BHMACHO)

• Massive/intermediate mass BH binaries
  - Formation history of central super massive BH

• Extreme (intermediate) mass-ratio inspirals (EMRI)
  - Probe of BH geometry
• **Inspiral phase** (large separation)
  Clean system (Cutler et al, PRL 70 2984(1993))
  Negligible effect of internal structure

Accurate prediction of the wave form is requested
- for detection
- for parameter extraction
- for precision test of general relativity
  (Berti et al, PRD 71:084025,2005)

• **Merging phase** - numerical relativity
  recent progress in handling BHs

• **Ringing tail** - quasi-normal oscillation of BH
Extreme mass ratio inspirals (EMRI)

- LISA sources 0.003-0.03Hz
  \[ M \sim 10^5 M_\odot - 5 \times 10^6 M_\odot \]
  \[ \rightarrow \text{merger to white dwarfs (} \mu = 0.6 M_\odot, \text{neutron stars (} \mu = 1.4 M_\odot, \text{BHs (} \mu = 10 M_\odot, \sim 100 M_\odot) \]

- Formation scenario
  - star cluster is formed
  - large angle scattering encounter put the body into a highly eccentric orbit
  - Capture and circularization due to gravitational radiation reaction \( \sim \text{last three years: eccentricity reduces} \ 1-e \rightarrow O(1) \)

- Event rate:
  a few \( \times 10^2 \) events for 3 year observation by LISA

\( \text{(Gair et al, CGQ 21 S1595 (2004))} \)
\( \text{(Amaro-Seoane et al, astro-ph/0703495)} \)
- $\mu \ll M$ → Radiation reaction is weak
- Large number of cycles $N$ before plunge in the strong field region

Roughly speaking, difference in the number of cycle $\Delta N > 1$ is detectable.

- High-precision determination of orbital parameters
- maps of strong field region of spacetime
  - Central BH will be rotating: $a \sim 0.9M$
Probably clean system


assuming almost spherical accretion (ADAF)

\[ t_{df} = \frac{v_{rel}^3}{4\pi \log \Lambda G^2 m_{satellite} \rho} \]

\[ \approx 4.5 \times 10^{12} \frac{M}{10^6 M_\odot} \left( \frac{\mu}{10 M_\odot} \right)^{-1} \left( \frac{\dot{m}}{10^{-2} \dot{M}_{Edd}} \right)^{-1} \text{yr} \]

Frequency shift due to interaction

\[ \frac{\Delta f}{f} \approx \frac{T_{obs}}{t_{df}} \approx 1 \text{yr} \]

Change in number of cycles

\[ \Delta N \approx \Delta f T_{obs} \approx f T_{obs} \frac{T_{obs}}{t_{df}} \]
Theoretical prediction of Wave form

Template in Fourier space

\[ h(f) \approx Af^{-7/6}e^{i\Psi(f)} \]

\[ A = \frac{1}{\sqrt{20\pi^3}} \frac{M^{5/6}}{D_L}, \quad M = \mu^{3/5} M^{2/5}, \quad \eta = \frac{\mu}{M} \]

\[ \Psi = 2\pi f t_c - \phi_c + \frac{3}{128} (\pi M f)^{-5/3} \left[ 1 + \frac{20}{9} \left( \frac{743}{331} + \frac{11}{4} \eta \right) u^{2/3} -(16\pi - \beta) u + \cdots \right] \]

\[ u \equiv \pi M f = O(v^3) \]

1PN 1.5PN for quasi-circular orbit

We know how higher expansion proceeds.

- Only for detection, higher order template may not be necessary?

We need higher order accurate template for precise measurement of parameters (or test of GR).

c.f. observational error in parameter estimate signal to noise ratio
Test of GR

Effect of modified gravity theory

Scalar-tensor type

\[ \Psi = \ldots + \frac{3}{128} (\pi M f)^{-5/3} \left[ \alpha u^{-2/3} + 1 + \left( \frac{3715}{756} + \frac{55}{9} \eta + \frac{128}{3} \eta \beta_g \right) u^{2/3} - (16\pi - \beta) u + \ldots \right] \]

\[ \alpha \propto \frac{1}{\omega_{BD}} \]

Dipole radiation = - 1 PN

Mass of graviton

\[ \beta_g = \frac{\pi^2 M}{\lambda_g^2} \int a^2 d\eta \]

\[ u = \pi M f = O(v^3) \]

Current constraint on dipole radiation:

\[ \omega_{BD} > 140, \ (600) \]

4U 1820-30 (NS-WD in NGC6624)


Constraint from future observation:

LISA– 10^7M_BH+10^7M_BH:
graviton compton wavelength
\[ \lambda_g > 1 \text{kpc} \]
(Berti & Will, PRD71 084025(2005))

LISA– 1.4M_NS+400M_BH: \[ \omega_{BD} > 2 \square 10^4 \]
(Berti & Will, PRD71 084025(2005))

Decigo–1.4M_NS+10M_BH : \[ \omega_{BD} > 5 \square 10^9 ? \]
Black hole perturbation

\[ G^{\mu\nu} [g] = 8\pi GT^{\mu\nu} \]

\[ g_{\mu\nu} = g_{\mu\nu}^{BH} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \cdots \]

✧ \( M \square \mu \)

✧ \( v/c \) can be \( O(1) \)

Linear perturbation

\[ \delta G^{\mu\nu} [h^{(1)}] = 8\pi G T^{(1)\mu\nu} \]

\[ L \psi^{(1)} = 4\pi \sqrt{-g} T^{(1)} \]: master equation

Regge-Wheeler formalism (Schwarzschild)

Teukolsky formalism (Kerr)

Mano-Takasugi-Suzuki’s method (systematic PN expansion)
Teukolsky formalism

Teukolsky equation

\[ L \psi = 4\pi \sqrt{-g} T \]

\[ T = \tau_{\mu\nu} T^{\mu\nu} \]

\(2\text{nd}\) order differential operator

Projected Weyl curvature

First we solve homogeneous equation

\[ L \Omega = 0 \]

\[ \Omega = \sum_{\Lambda} R_{\Lambda}(r) Y_{\Lambda}(\theta, \varphi) e^{-i\omega t} \]

\[ \Lambda \equiv \ell, m, \omega \]

Angular harmonic function

\[ \left[ \partial_r^2 + \cdots \right] R(r) = 0 \]

Construct solution using Green fn. method.

\[ \psi = \sum_{\Lambda} \Omega_{\Lambda}^{up}(x) Z_{\Lambda} \approx \sum_{\Lambda} \Omega_{\Lambda}^{up}(x) \frac{1}{W_{\Lambda}} \int \sqrt{-g} d^4 x' \ R^{in}_{\Lambda}(r') \bar{Y}_{\Lambda}(\theta', \varphi') e^{i\omega t} T(x') + \cdots \]

Wronskian

\[ W_{\Lambda} \approx R^{up}_{\Lambda} \partial_r R^{in}_{\Lambda} \]

At \( r \to \infty \)

\[ \psi \sim \frac{1}{2} \left( \ddot{h}_+ - i \dot{h}_x \right) \]

\[ \frac{dE}{dt} = -\sum_{\Lambda} |Z_{\Lambda}|^2 \quad \text{: energy loss rate} \]

\[ \frac{dL_z}{dt} = -\sum_{\Lambda} \frac{m}{\omega} |Z_{\Lambda}|^2 \quad \text{: angular momentum loss rate} \]
Leading order wave form

Energy balance argument is sufficient.

\[
\frac{dE_{GW}}{dt} = - \frac{dE_{orbit}}{dt}
\]

Wave form \( f \) for quasi-circular orbits, for example.

\[
\frac{df}{dt} = \frac{dE_{orbit}}{dt} / \frac{dE_{orbit}}{df}
\]

leading order

\[
\frac{dE_{orbit}}{dt} = 0 + O(\mu) + O(\mu^2)
\]

self-force effect

\[
\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)
\]
Radiation reaction for General orbits in Kerr black hole background

Radiation reaction to the Carter constant

**Schwarzschild** “constants of motion” $E, L_i \leftrightarrow$ Killing vector
Conserved current for GW corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^\mu t_{\mu\nu}^{(GW)} \xi^\nu$$

$$\dot{E}_{\text{orbit}} = -\dot{E}_{\text{gw}}$$
In total, conservation law holds.

**Kerr** conserved quantities $E, L_z \leftrightarrow$ Killing vector

$Q \leftrightarrow$ Killing vector

We need to directly evaluate the self-force acting on the particle, but it is divergent in a naïve sense.
Adiabatic approximation for $Q$, which differs from energy balance argument.

- orbital period $\ll$ timescale of radiation reaction
- It was proven that we can compute the self-force using the radiative field, instead of the retarded field, to calculated the long time average of $\dot{E}, \dot{L}_z, \dot{Q}$.

$$h_{\mu\nu}^{(rad)} = \left[h_{\mu\nu}^{(ret)} - h_{\mu\nu}^{(adv)} \right]/2$$  : radiative field

At the lowest order, we assume that the trajectory of a particle is given by a geodesic specified by $E, L_z, Q$.

$$\langle \dot{Q} \rangle = \frac{1}{\mu} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\tau \frac{\partial Q}{\partial u^\mu} F^\alpha \left[ h_{\mu\nu}^{(rad)} \right]$$

Radiative field is not divergent at the location of the particle.  \hspace{1cm} Regularization of the self-force is unnecessary!
Simplified $dQ/dt$ formula

(Sago, Tanaka, Hikida, Nakano, Prog. Theor. Phys. 114 509('05))

- Self-force $f^\alpha$ is explicitly expressed in terms of $h_{\mu\nu}$ as
  \[ f^\alpha = - \frac{1}{2} \left( g^{\alpha\beta} + u^\alpha u^\beta \right) \left( h_{\beta\gamma;\delta} + h_{\beta\delta;\gamma} - h_{\gamma\delta;\beta} \right) u^\gamma u^\delta \]

\[ \frac{dQ}{d\tau} = 2 K^\nu_\mu u^\mu f^\nu \]

Killing tensor associated with $Q$

$Q \equiv K^\nu_\mu u^\mu u^\nu$

$\begin{align*}
  h_{\mu\nu} &= \tau^{\mu\nu} \Psi \\
  \text{Complicated operation is necessary for metric reconstruction from the master variable.}
\end{align*}$

- We arrived at an extremely simple formula:
  \[ \left< \frac{dQ}{dt} \right> = 2 \left< \frac{r^2 + a^2}{\Delta} P(r) \right> \left< \frac{dE}{dt} \right> - 2 \left< \frac{aP(r)}{\Delta} \right> \left< \frac{dL}{dt} \right> + 2 \sum_{l,m,\omega} \frac{n_r \Omega_r}{\omega} |Z_{l,m,\omega}|^2 \]

Only discrete Fourier components exist

$\omega = \omega_m^{n_r, n_\theta} \approx \left( m\Omega_\varphi + n_r \Omega_r + n_\theta \Omega_\theta \right)$

$\begin{align*}
P(r) &= E \left( r^2 + a^2 \right) - aL \\
\Delta &= r^2 - 2Mr + a^2
\end{align*}$
Use of systematic PN expansion of BH perturbation.
Small eccentricity expansion
General inclination

\[ \left\langle \frac{dC}{dt} \right\rangle = -\frac{64}{5} \left( \frac{\mu}{M^2} \right) M^2 v^6 (1 - e^2)^{3/2} (1 - Y^2) \left[ \left( 1 + \frac{7}{8} e^2 \right) 
- \left( \frac{743}{336} - \frac{23}{42} e^2 \right) v^2 
- \left( \frac{85Y}{8} + \frac{211Y}{8} e^2 \right) q v^3 
+ \left( 4 + \frac{97}{8} e^2 \right) \pi v^3 
- \left( \frac{129193}{18144} + \frac{84035}{1728} e^2 \right) v^4 
- \left( \frac{329}{96} - \frac{53Y^2}{8} + \left\{ \frac{929}{96} - \frac{163Y^2}{8} \right\} e^2 \right) q^2 v^4 
+ \left( \frac{2553Y}{224} - \frac{553Y}{192} e^2 \right) q v^5 
- \left( \frac{4159}{672} + \frac{21229}{1344} e^2 \right) \pi v^5 \right] \]

(Ganz, Hikida, Nakano, Sago, Tanaka, Prog. Theor. Phys. (’07))
Summary

Among various sources of GWs, E(I)MRI is the best for the test of GR.

For high-precision test of GR, we need accurate theoretical prediction of the wave form.

Adiabatic radiation reaction for the Carter constant has been computed.

\[
\frac{dE_{\text{orbit}}}{dt} = 0 + O(\mu) + O(\mu^2)
\]

\[
\frac{dE_{\text{orbit}}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)
\]

Direct computation of the self-force at \(O(\mu)\) is also almost ready in principle.
However, to go to the second order, we also need to evaluate the second order self-force.